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ABSTRACT

In October 1975 a conference was convened in Euclid, Ohio, by the Basic Skills Group of the National Institute of Education (NIE). Thirty-three participants presented position papers addressing two major questions: (1) What are basic mathematical skills and learning? (2) What are the major problems related to children's acquisition of basic mathematical skills and learning, and what role should the National Institute of Education play in addressing these problems? These papers are presented in this volume. Conferees organized into small working groups during the conference. Conference participants included representatives from public school systems; university faculties of education, mathematics, and psychology; educational research and development organizations; textbook publishers; the National Science Foundation; and NIE. The papers range from broad philosophical discussions of the issues to specific detailing of objectives and problems. (SD)

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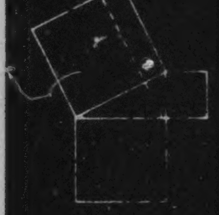

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conference on basic mathematical skills and learning

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THE NIE CONFERENCE ON
BASIC MATHEMATICAL SKILLS AND LEARNING

VOLUME I:
CONTRIBUTED POSITION PAPERS

OCTOBER 4-6, 1975
EUCLID, OHIO

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PREFACE

In October 1975, the Basic Skills Group of the National Institute of Education sponsored and organized a three-day Conference on Basic Mathematical Skills and Learning, held in Euclid, Ohio. The invited participants were asked to contribute short papers in response to the following questions:

1. What are basic mathematical skills and learning?

2. What are the major problems related to children's acquisition of basic mathematical skills and learning, and what role should the National Institute of Education play in addressing these problems?

Every participant except the two representing Federal agencies wrote a paper. After the conference, the authors had the opportunity to alter their papers as they wished, and the papers were subsequently edited to conform to Government Printing Office style. The present volume contains the papers in their final form, ordered alphabetically by the authors' last names. (If a paper was written by two or more persons, the conference participant's name appears first, followed by the remaining authors' names in alphabetical order.)

Some of the papers refer to a memorandum from NIE about the word "skills." The memorandum was written simply to assure the participants that NIE's interest in "basic mathematical skills" goes beyond the ability to perform written computational algorithms rapidly.

Much of the conference was spent in small working groups. A separate volume contains the reports of these groups, together with a description of the organization of the conference, and some summarizing remarks by Peter Hilton and Gerald Rising, who acted as cochairmen of the conference.

We would like to thank Ms. Nitina Chavan of Nero & Associates, who was responsible for coordination and arrangements before, during, and after the conference. And we are certainly grateful to the conference participants, whose ideas and experience are reflected in the papers of this volume.

Edward Esty
NIE Associate, Learning Division

A VIEW FROM WITHIN THE CLASSROOM

Charles E. Allen

It was with great pride and honor that I tackled this paper. The conference planners are to be commended for considering a classroom teacher who meets daily with students at the secondary level.

What are the basic mathematical skills and learning? My dictionary and I came up with the following rephrasing of the conference topic. What are the essential high states of performance or expertness in mathematics that a person should be able to easily or expertly apply? What essential knowledge of mathematics will permit this high state of performance or expertness?

Come into my Algebra S,1A Class today and solve this problem:

$$\begin{array}{r} 8\frac{1}{4} \\ 2\frac{1}{2} \\ 3\frac{1}{5} \\ 4\frac{4}{8} \\ 5\frac{4}{10} \\ +\ 6\frac{4}{16} \\ \hline \end{array}$$

Go ahead and solve the problem. You can never appreciate the view from within unless you get involved.

"Algebra S,1A" indicates that these students could not master the year of algebra in one year. They will be permitted to study it for a 1½ to 2-year period. Basic algebraic skills don't come easy to these students!

Many people would see solving this problem as a basic skill. To the student, it is an application of many basic skills. The breakdown of the skills involved shows that the student has more than 34 chances to miss this problem. The student has more than 34 chances to fail.

$$(1)\ \begin{array}{r} 8\frac{1}{4} \\ 2\frac{1}{2} \\ 3\frac{1}{5} \\ 4\frac{4}{8} \\ 5\frac{4}{10} \\ +\ 6\frac{4}{16} \\ \hline \end{array} \qquad (2)\ \begin{array}{r} 8 \\ 2 \\ 3 \\ 4 \\ 5 \\ +\ 6 \\ \hline \end{array} \qquad (3)\ \begin{array}{r} 8 \\ +\ 2 \\ \hline \end{array} \quad (4)\ \begin{array}{r} 10 \\ +\ 3 \\ \hline \end{array} \quad (5)\ \begin{array}{r} 13 \\ +\ 4 \\ \hline \end{array} \quad (6)\ \begin{array}{r} 17 \\ +\ 5 \\ \hline \end{array} \quad (7)\ \begin{array}{r} 22 \\ +\ 6 \\ \hline \end{array}$$

$$(8)\ \begin{array}{r} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{5} \\ \frac{4}{8} \\ \frac{4}{10} \\ +\ \frac{4}{16} \\ \hline \end{array}$$

$$(9)\left|\underline{\ 4, 2, 5, 8, 10, 16\ }\right.$$

(10) L.C.D. = 2x2x2x5x1x1x1x1x1x2 = ?

(11) $\frac{1}{4} = \frac{?}{80}$

$\frac{1}{2} = \frac{?}{80}$

$\frac{1}{5} = \frac{?}{80}$

$\frac{4}{8} = \frac{?}{80}$

$\frac{4}{10} = \frac{?}{80}$

$\frac{4}{16} = \frac{?}{80}$

(12) $4\overline{)80}$ (13) $2\overline{)80}$ (14) $5\overline{)80}$ (15) $8\overline{)80}$

(16) $10\overline{)80}$ (17) $16\overline{)80}$

(18) $20 \times 1 = ?$ (19) $40 \times 1 = ?$ (20) $16 \times 1 = ?$

(21) $10 \times 4 = ?$ (22) $8 \times 4 = ?$ (23) $5 \times 4 = ?$

(24)
$$\begin{array}{r} \frac{20}{80} \\ \frac{40}{80} \\ \frac{16}{80} \\ \frac{40}{80} \\ \frac{32}{80} \\ + \frac{20}{80} \\ \hline \end{array}$$

(25) $\begin{array}{r} 20 \\ +40 \\ \hline \end{array}$ (26) $\begin{array}{r} 60 \\ +16 \\ \hline \end{array}$ (27) $\begin{array}{r} 76 \\ +40 \\ \hline \end{array}$ (28) $\begin{array}{r} 116 \\ +32 \\ \hline \end{array}$ (29) $\begin{array}{r} 148 \\ +20 \\ \hline \end{array}$

(30) $\frac{168}{80} = ?$ (31) $80\overline{)168}$ (32) $2\,\frac{8}{80} = ?$

(33) $\frac{8}{80} = ?$ (34) $\begin{array}{r} 28 \\ +\;2\,\frac{1}{10} \\ \hline \end{array}$

Step (1) presents our first basic skill which is to read the problem. Just reading 8¼ as eight and one-fourth alerts the student to the fact that the parts may be separated and added separately. Many of my students have difficulty with the basic skill of reading the problem. It is not just in my classroom, either. Ask Mary Ann Hater, Robert B. Kane, and Mary Ann Byrne (1974) about Helping Children Read Mathematics.

Looking at our problem, Max S. Bell (1974) would see three of his skills that "Everyman" really needs from school mathematics. This one problem challenges the student to make efficient and informed use of computational algorithms, to make flexible selection and use of appropriate elements from equivalence classes, and to make rapid and accurate calculation with one and two digit numbers.

Suppose that E. L. Edwards, Eugene D. Nichols, and Glyn H. Sharpe (1972) dropped into my class today. They would smile approvingly when they saw the students using whole numbers in problem solving, writing equivalent fractions for given fractions, using standard algorithms for the operations of arithmetic or positive rational numbers, and solving addition problems involving fractions. Of course, these students are in the group that view mathematics as a tool for effective citizenship and personal living. They just want enough to "barely get by."

If only my students were skilled at the mathematical competency of expressing a rational number using decimal notation. Can't you see Gene Nichols going to the board with that glint in his eyes as he shows them this method of solving the problem?

$$\begin{array}{rl} 8\frac{1}{4} & = 8.25 \\ 2\frac{1}{2} & = 2.50 \\ 3\frac{1}{5} & = 3.20 \\ 4\frac{4}{8} & = 4.50 \\ 5\frac{4}{10} & = 5.40 \\ +\ 6\frac{4}{16} & = 6.25 \\ \hline & 30.10 = 30.1 \end{array}$$

Please do not begrudge me the three pages used with this one problem, because I sincerely feel that this problem best indicates our need and most accurately presents the challenge that confronts us at this conference.

Daily, we lose students at each of the 34 steps in solving this problem. The students become the ones that Mitchell Lazarus (1974) described as seeing mathematics as an obstacle in school and as irrelevant in adulthood. These students will not reach the level of mathematical sophistication we have reached. They will not be permitted to make the careless mistakes that some of you made with this problem. They may not be provided with calculators that you find so convenient to use. Many of them will not even be permitted to use the times table. Yet, can we say that these students will not become enlightened citizens? Can we say that they will not get by in our contemporary society?

I hope that we will not use this conference only to produce yet another list of basic skills. There are some serious problems facing teachers on the firing line, daily. NIE must make a study of the current standard tests being administered. They must carefully analyze the questions, the responses, the detractors, the stems, and many of the variables that influence test results. They must also consider the conditions under which tests are administered. Many teachers need to learn how to administer tests. Many students need to learn how to take tests.

I regret that I cannot give you specific examples, but I have no permission from the group holding the copyright to do so. Tell me why 84 percent of the sixth graders tested can tell how many cents a certain number of quarters, dimes, nickels, and pennies amount to when only 27 percent of the same group correctly added three numbers involving pounds and ounces? Forty-five percent of these sixth graders were able to subtract denominate numbers involving feet and inches. Only 20 percent could subtract denominate numbers involving weeks and days. Can you believe this last one?

When 90 percent correctly respond to a multiplication problem involving multiplying two lengths in inches to get a certain number of square inches, I did not know whether or not to attribute it to the excellent work we have done with multiplication.

What percent of these same sixth graders do you think could solve this problem correctly?

$$6\overline{)\,9 \text{ feet } 6 \text{ inches}}$$

() 1 foot 1 1/2 inches

() 1 foot 6 inches

() 1 foot 7 inches

() 1 foot 9 inches

What percent of their big sisters and brothers in the twelfth grade could solve the same problem? The answer appears in another part of this report. Your basic skill is to find it.[1]

Then there are the types of questions that NIE will not need to consider beyond the reading of the question. In one such problem, the students were asked to read a bar graph. This, they proved that they could do. Then the following question asked them what fractional part of the total enrollment was the attendance on Thursday. What skill is being tested here? What skill is the student failing?

Regardless of copyright violations, I must show you this one:

> "The perimeter of an equilateral triangle is 15 inches. The length of each side of a square is 2 inches more than twice the length of each side of the triangle. What is the perimeter of the square?"

The item is to test measurement skills: Comprehension and application. After seeing how you did on the mixed number problem presented at the beginning of this paper, I will not tempt you with the answer to this problem.

NIE must give strong consideration to identifying the typical student, his or her abilities and needs. NIE must identify the typical teacher, his or her strengths and weaknesses. NIE must identify the typical texts and their strengths. NIE must identify the typical teacher-training institution and its strengths and weaknesses. Then NIE must come forth with more than one approach that may be used to answer the question, "Why Johnny can't add?" NIE must get all of these groups to pull together!

Some of the people mentioned earlier have made suggestions that NIE could use in its task. Hater, Kane, and Byrne (1974) started by having the teachers list the most vexing problems that confronted them. Bell (1974) described the school mathematics experience as a clear failure for the majority of the students. He further identified the key to an adequate mathematics experience as lying in the years before high school. Bell looks to Piaget and others for a routing to a better early school mathematics experience. Edwards (1972) and his gang were "right on" with their three basic views of mathematics. NIE needs now to identify the percent of the population that traditionally falls within each of these three groups. This would permit us to reallocate our efforts and energies. We do spend a lot of time looking for the Gauss out of 1 million students. Lazarus (1974) mentions several of the approaches that Project One is using to depart from the traditional approach to teaching mathematics.

Since you are in my classroom today, I would like to relate some other concerns that teachers have. Do you have the answers to our questions?

---

[1] Of the sixth graders, only 13 percent correctly answered the same question that 42 of their twelfth grade brothers and sisters answered.

1. How are our bright mathematics students doing on the advanced placement examination in calculus?

2. Is it necessary for our liberal arts students to struggle through geometry?

3. Why can't we generate more excitement about mathematics clubs?

4. Why do students tire so soon with calculators, computers, and other innovations?

5. Why are there no really good textbooks written for the masses of students who take only nonacademic classes in mathematics?

6. Aren't the average scores of the college-bound students on the SAT low?

7. Why can't a test be designed that will measure to what extent the students have learned what was taught? Why must tests always measure what some expert thinks should have been taught?

8. Why can't some national movement be started that will focus on the teacher, his or her needs and desires?

Boey (1975) shared some of her classroom gems with Frontier Magazine recently. One youngster had written the following letter after his study of aviation.

> "I want to be a pilot in the Army Corp when I grow up. Could you please tell me how good I have to be in math, because if I have to be real good, you better not count on me to join up. Good bye."

When Dr. Haim G. Ginott (1972) was a young teacher he wrote the following passage.

> "I have come to a frightening conclusion. I am the decisive element in the classroom. It is my personal approach that creates the climate. It is my daily mood that makes the weather. As a teacher I possess tremendous power to make a child's life miserable or joyous. I can be a tool of torture or an instrument of inspiration. I can humiliate or humor, hurt or heal. In all situations it is my response that decides whether a crisis will be escalated or de-escalated, and a child humanized or de-humanized."

Dr. Ginott discussed attitudes expressed in skills as being what counts in education. One teacher that he quotes, had this to say.

> "I already know what a child needs. I know it by heart. He needs to be accepted, respected, liked, and trusted; encouraged, supported, activated, and amused; able to explore, experiment, and achieve.

> Damn it! He needs too much. All I lack is Solomon's wisdom, Freud's insight, Einstein's knowledge, and Florence Nightingale's dedication."

Dr. Ginott related a mother's testimony after she had visited her son's classes during an open school week visit. Her trip through the classes left her stunned.

> "I went home, stunned by what I had seen. Thoughts ran through my head. No one had smiled. In the whole time, no one had smiled...No teacher showed delight in children. No teacher had bothered to motivate. Motivation is not just a procedural item taught in teacher's college...It is the process that prepares a child for loving. All that we call education was conceived in love. It was because man so loved the shell that he gave it a name. It was his way of getting closer to it, paying homage to it, of summoning it up in his thoughts even when the creature itself was absent. The act of naming is the act of loving - be it a decimal point, a part of speech, or a shell. And the act of loving requires preparation - warmth, caring, ease, sensitivity, tenderness, skill. What I had witnessed had nothing of this. It was more like a sadistic attempt at forcible penetration - a raping of children. And still we demand that the children respond. There is only one proper response to rape. For the woman it is the closing of her legs; for a child, the closing of his mind."

This conference is starting to solve a herculean task. There are parents out there crying for us to teach their children to make change, to multiply, and to balance the checkbook. There are administrators out there crying that we raise the mean, median, mode, or whatever that number in the middle is called. There are newspaper editors out there crying every 5 years that the "new math" does not work. There are vocational teachers out there crying that we teach the kid how to locate one-eighth of an inch on the ruler. There are boards of education out there who would hire or fire us on the basis of what the students do on unrealistic tests. There are legislators out there trying to tell the publishers what to put into and what to take out of textbooks. There are authors out there blaming us because "Johnny can't add." There are Johnnies out there that cannot even count. There are mathematics teachers out there that do not even know where to start in teaching a student how to add. Please do not use this conference to debate whether deriving the quadratic formula is a skill or not.

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BASIC MATHEMATICAL SKILLS

J. M. Armstrong

For most Americans, the main reason for learning mathematics is to acquire tools for solving real, everyday problems. This requires the mental skills of reasoning, problem solving, and analytical thinking. It includes the identification and formulation of specific problems, the solution of a problem translated into a mathematical form, computations, comparison of the results with previous observations, and the drawing of appropriate conclusions. The appropriate use of mathematics in these situations is critical.

Most people, at least by the time they finish high school, should have the skills I have listed below. I have tried to include in it the ideas and skills of mathematics needed to make intelligent decisions about real world problems and to provide a basis for more specialized learning of mathematics.

Numbers and Their Uses

1. Counting.

2. Ordering.

3. Notation (including decimal notation and powers of 10).

4. Measuring.

5. Computation (quick recall of basic number facts and efficient and accurate use of the four basic operations with whole numbers, integers, and rational numbers).

Variables and Relationships

1. Ratio, proportion, and percent (including scaling and mapping).

2. Relations (such as greater than, less than, equal to, parallel, perpendicular, and so on).

3. Setting up and solving simple algebraic equations.

4. Knowledge and appropriate use of formulas ($d = rt$, $A = lw$).

5. Exponential and arithmetic relationships (especially as exponential functions apply to such problems as population growth).

Measurement

Selection of:

1. Relevant attributes (what characteristics to measure).
2. Appropriate unit (kind and size including both English and metric).
3. Degree of precision.
4. Appropriate instrument.

Techniques of:

1. Using measuring instruments.

2. Estimation and approximation (especially orders of magnitude, "ball park" estimates, and quick approximations, using very large numbers; judging the reasonableness of an answer).

Logic

1. Identifying fallacies in arguments (especially in advertising).
2. Constructing and following logical arguments.

Geometry

1. Projection from three to two dimensions.
2. Similarity and congruence of triangles.
3. Position by coordinates.
4. Determining area and perimeter of common geometric shapes.

Probability and Statistics

1. Exploring data for patterns.

2. Reading and interpreting graphs, tables, and charts.

3. Representative samping from populations (its use in making surveys such as the Gallup Poll).

4. Descriptive measures -- mean, median, mode, standard deviation.

Computer Literacy

1. How computers process information.

2. Applications and impact on society.

Although not specifically mentioned here, consumer mathematics is an important application of basic mathematical skills. Solving consumer problems requires all the previously listed skills. When the results of the first mathematics assessment were recently released by National Assessment of Educational Progress, we learned that eight out of ten young adults (age 26-35) cannot balance a checkbook, and that half are unable to make out a simple income tax form without making a mistake. Considering the presence of double-digit inflation, these are startling results. The implication seems clear. Mathematics needs to be taught using more real world applications.

Problems in Learning Basic Mathematical Skills

This section briefly discusses three important and broad areas of concern: (1) basic research on how children learn, (2) teacher education, and (3) testing and evaluating programs and children.

How do children learn? The literature indicates mathematical development in children is a complex series of stages of learning. Little research has been done to identify these stages and determine which are common or expected at different ages or levels of maturation. The relationship between the Piagetian model of learning and development needs to be verified for learning mathematics.

Techniques of interviewing children need to be developed so we can identify children's thinking patterns. Children should be interviewed on a one-to-one basis, to determine these modes of thinking in relation to learning mathematical concepts and problem solving. There is evidence that some problem-solving skills can be identified, generalized, and taught.

Individuals or centers should be encouraged to do basic research on how children acquire mathematical understanding and what strategies and skills they use when solving nonroutine problems. These findings should then be incorporated into the curriculum.

The results of basic research on how children acquire mathematical skills also needs to be incorporated in teacher education programs. Teachers play the most critical role in the learning experiences of children. Effective teacher education programs (both preservice and inservice) are essential if children are going to learn more than computation in the elementary school.

The prospective teacher needs more direct experience with children prior to teaching in the classroom. Methods courses should be augmented with real classroom experience during preservice training. The methods course should be taught in small groups, using the manipulatives most often found in mathematics laboratories.

Children in the elementary shcool need much more instruction using physical objects and manipulatives. Manipulatives are especially valuable when teaching concepts such as counting, place value, and fractions. If teachers are trained using these manipulatives, they are most likely to use them in the classroom.

Presently, undergraduates training to be elementary school teachers need to acquire expertise in all areas of the elementary school curriculum -- language arts, reading, mathematics, science, etc. It seems impossible for prospective teachers to learn every skill and technique they need to know before entering the classroom. Two recommendations are made. First, that teachers be taught how and where to continue their learning/training; and second, that in-service training be more readily available to the teacher.

Teachers should be encouraged to continue their education through workshops, professional organizations, seminars, and university course work. School districts should have resource centers staffed by competent personnel. These centers would provide workshops, material from various sources (curriculum projects, publishers, research) and expert advice to help the teacher with his/her problems.

The third major problem associated with a child's learning of mathematics is related to present testing and evaluation efforts in mathematics education. Decisions regarding the success of curriculum projects are rarely based on tests sensitive to the goals of the program. Often, batteries of standardized tests are used. These norm-referenced tests are developed to rank students' achievement in relation to each other, not to measure specific goals or objectives.

An integral part of a curriculum project should be the development of a wide variety of appropriate instruments for program evaluation.

Another problem related to evaluation is student testing. A center needs to be organized to collect and develop instruments and new techniques to assess important objectives in mathematics. Tests to measure lower level skills in mathematics (i.e., computation) are abundant and adequate. More sophisticated tests need to be written for high order abilities such as problem solving, modeling, and decisionmaking.

A dearth of information exists for tests which reliably measure attitudes toward mathematics. Test development could also include interview techniques and observation schedules which could be an integral part of the basic research into the patterns of thinking in younger children.

Last, data from national evaluation efforts in mathematics need to be analyzed carefully for their learning and curriculum implications. The results on computation from the first assessment of mathematics by NAEP are readily available for such interpretations. This assessment included data on exercises administered in both group and individual situations. Individually administered exercises could be analyzed in terms of the process used to solve a particular problem. Open-ended group administered exercises could be analyzed in the same way, when the respondent showed his work.

Recommendations to NIE

1. Research projects related to determining how children learn mathematics are needed (like the Project for the Mathematical Development of Children, funded by NSF).

2. Basic research on problem-solving strategies and techniques is urgently needed, especially for younger children.

3. Development and dissemination of curriculum projects designed to improve inservice teacher training is needed.

4. Mathematics resource centers need to be organized at the local level for inservice teacher education.

5. Requests to fund a curriculum project should be required to include plans for a thorough evaluation of the project.

6. A wide variety of sensitive, valid, and reliable tests of mathematical learning need to be developed.

7. National data on mathematics learning and achievement need to be analyzed in terms of curriculum implications.

BASIC SKILLS IN MATHEMATICS

E. G. Begle

Preface

My discussion of the basic skills in mathematics will depend, in part, on some concepts developed by SMSG for the National Longitudinal Study of Mathematical Abilities. We made very little use of standardized tests in that study. For one thing it was impossible to find tests that would adequately touch on all of the objectives of the great variety of textbooks that were used in the study. Second, and most important, when a student does poorly on a standardized test, no information is provided as to where the difficulties lie.

For the development of our own tests, we used the following conceptual framework. We thought of mathematical activities as taking place at three cognitive levels. The first is that of rote computation. An activity at this level consists in carrying out, accurately, previously learned algorithms.

The second cognitive level we labeled comprehension. Activities at this level consist of translating concepts from any of the verbal, symbolic, or iconic forms to any of the others; understanding and explaining why algorithms work; understanding and explaining mathematical principles; and recognizing examples of specific concepts in mathematical and in real life situations.

The last cognitive level is that of problem solving. Activities at this level range from the recognition of familiar problem types, followed by the application of the appropriate algorithm, to the analysis of completely unfamiliar problems.[1]

In developing a test battery (two batteries were administered each year for 5 years), we started with a matrix of the following type:

	Arithmetic/ Algebra	Geometry	Other (e.g., probability, functions, etc.)
Computation			
Comprehension			
Problem solving			

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[1] Actually, this last cognitive level was split into two: applications, the solution of familiar problems; and analysis, the solution of unfamiliar problems. However, the distinction turned out to be somewhat ambiguous and, for our present purposes, they are combined into one.

We refined the column headings to be appropriate to the grade level being tested. We then developed relatively short homogeneous tests for each of a number of the cells in the matrix. The totality of these tests constituted the test battery, but we recorded and analyzed students' scores on the individual tests separately.

Analysis of a large number of such test batteries shows clearly that these three cognitive levels are empirically as well as conceptually different. The abilities needed to function effectively at one level are not the same as those needed for another level. These three kinds of abilities are of course intercorrelated, but the correlations are relatively modest.

Basic Skills in Mathematics

I consider basic skills as those that are likely to be useful to the average citizen in his or her everyday life. If this definition is accepted, then it is clear that some of the basic skills will be at the cognitive level of problem solving. Certainly the average person is frequently faced with problems which, though not necessarily very complex, nevertheless require mathematics for their solution.

In addition, some of the basic skills will be at the cognitive level of understanding. Mathematical concepts and principles are often useful in describing and understanding the physical and social worlds we live in. Even more important, an understanding of mathematical concepts and principles is needed for problem solving.

Finally, some of the basic skills will be at the cognitive level of computation. Skill should be developed at this level to whatever extent is necessary for the other two cognitive levels.

Whatever specific set of basic skills in mathematics is finally agreed upon, all three cognitive levels must be represented.

Mathematical Content of Basic Skills

In 1965, the School Mathematics Study Group had just about finished its task of providing, for each grade level from K through 12, an example of a textbook in which the emphasis was much greater on the understanding of mathematical concepts and principles than was the case in the textbooks in use during the 1950's. Although information from the National Longitudinal Study was just beginning to come in, the SMSG Advisory Board felt that it was an appropriate time to review the situation and to decide whether any further curriculum development was needed.

After lengthy discussions the board decided that a second round of curriculum development, at the junior high school level, was needed. Two major arguments led to this decision. The first was that the original SMSG texts had not tried to change the customary (in the United States) grade placement of topics (e.g., arithmetic and informal geometry in grades 7 and 8, algebra in grade 9, formal geometry in grade 10, etc.). It was argued that freedom to bring in some algebraic and geometric concepts earlier and to postpone others until later could lead to a more efficient curriculum in which the various branches of mathematics provided more mutual support for each other.

Second, it had become clear that certain mathematical topics had in recent years become so important in our culture that at least an introduction to these ideas should be provided to every student. Chief among these topics were probability (and statistics) and the basic ideas of computer science. Because all students in this country are required to take mathematics at least through the ninth grade it was felt that an introduction to these two major topics ought to be included in the junior high school program.

Finally, SMSG had insisted from the very beginning that mathematics was included in the compulsory curriculum because of its utility, and claimed that an understanding of mathematical concepts and principles would be of more help to the average citizen in coping with our complex technological culture than the mere possession of certain computational skills. Nevertheless, despite the vigorous efforts of the original SMSG writing teams, the SMSG Advisory Board was not convinced that enough had been done to make clear, to the average student, the relevance of mathematics to real world problems and that one more attack on this problem needed to be made.

In March 1966, a conference of classroom teachers, pure mathematicians, applied mathematicians, physicists, and engineers, met for a week to draw up more detailed plans for such a curriculum. Two major recommendations came from this conference:

1. The attempt to make clear to students the relevance of mathematics to problems of the real world should be done by frequent consideration of mathematical models of significant and interesting problem situations in a variety of areas.

2. The curriculum for grades 7 through 9 should contain an introduction to those mathematical concepts, and only those concepts, which are important to the general citizen. This curriculum should be for all students, with those who are less able moving at a slower pace, and taking longer to complete it.

In connection with mathematical models, the conference stated that "the activity of constructing and analyzing mathematical models of scientific and life situations is the principal link between mathematics and the rest of civilization."

The conference listed some specific skills to be inculcated and specific insights to be presented:

A. Skills

1. Drawing of diagrams, tables, flow charts, etc., to describe situations.

2. Familiarity with useful symbols and functions, such as max, min, absolute value and, most especially, subscripts; some facility in naming variables and in using symbols to designate constants not yet known explicitly.

3. Ability to express mathematical or near mathematical ideas in reasonable sentences, aloud and in writing.

4. Rejection of some facts of life from models as being of lesser significance.

B. Insights

1. Recognition that exact answers are not always obtainable and that experimental calculations may be useful.

2. Recognition of limits of applicability of models even when exact answers are obtainable (see A 4).

3. Appreciation that not all useful insights are numerical.

4. Possibility of the same model form representing different situations.

As for the second recommendation, the process of curriculum-building that followed in the next few years indicated that some algebraic concepts could profitably be introduced in grades 7 and 8, while others could well be postponed until after grade 9. Similarly, some geometric concepts and principles (theorems) could well be introduced also in these grades, but that any discussion of axiomatics should be postponed until grade 10 or later.

It also turned out that it is perfectly feasible to introduce some ideas about flow charting, and some introductory concepts in probability, starting in grade 7.

The writers who constructed this new junior high school curriculum were required to provide, in the teacher's commentary for each chapter, two paragraphs. One spelled out, in general terms, the specific objectives of the chapter; the second explained why the topics in the chapter would be appropriate for the average citizen. These paragraphs form the text of section 2, titled "Minimum Goals for Mathematics Education," of SMSG Newsletter No. 38, August 1972.

What Can Be Done to Help to Improve the Teaching and Learning of the Basic Skills in Mathematics?

I will present my discussion under six general headings.

1. The goals of mathematics education. No matter what set of goals we agree on, some of them will easily be seen as basic while others will be ancillary. The question then arises as to how far we wish to carry these ancillary objectives. Thus, for example, we clearly want everyone to be able to handle personal financial accounting. A prerequisite for this is the ability to add whole numbers, but how much of this ability is really needed? Is it enough to be able to add correctly a column of four three-digit numbers, or should be ask everyone to be able to add correctly a column of ten six-digit numbers, or should we ask for something in between?

Answers to this kind of question would be exceedingly useful to teachers and to textbook writers. Answers can be obtained empirically and efforts to do so should be supported.

It would be helpful to have available a large number of criterion-referenced tests covering the basic mathematical skills. (These would, incidentally, spell out in detail what we mean by the basic skills.) Such tests are almost essential if teachers are going to be asked to take account of individual differences within their classrooms.

2. Teachers. Efforts to characterize the effective teacher have been attempted for the past 75 years. Practically nothing useful has been found. It would not seem wise to support any further efforts here.

3. The curriculum. There are questions about many of the variables under this heading whose answers would be helpful to classroom teachers, textbook writers, and mathematics educators in general. On many of these questions, enough empirical evidence has been gathered, through small experiments, to indicate that a larger and broader study would probably produce the sought for answers.

There are some variables in this list that the evidence to date suggests are of no value in teaching the basic skills, but for which a larger amount of information would be needed to settle the matter.

Finally, there are some questions for which we now have essentially no empirical information, but which are so important that they must be pursued. A prime example is the question of the usefulness of hand-held calculators in teaching the basic skills.

In general, a careful review of the literature is needed to plan a program of work in this area.

4. Teaching procedures. During the past 15 years, more than 1,000 publications (journal articles, papers delivered at AERA or NCTM meetings, dissertations, etc.) have appeared containing empirical information about the effects of various pedagogical procedures on student achievement in mathematics. Unfortunately, very little advice to classroom teachers can be based with any confidence on this research. When the studies on any one particular aspect of teaching mathematics are put together, the information is usually found to be spotty and incomplete.

It would clearly be worthwhile to undertake a program of research on various important aspects of mathematics teaching, in order to be able to provide convincing evidence as to the value (or lack of value) of various pedagogical procedures, convincing not only to the research community but also to classroom teachers.

Homework. A few examples may be suggested. The research that has been done on the value of homework in connection with mathematics teaching indicates quite clearly that further research on this topic is not recommended: Homework seems to contribute very little toward achievement in mathematics.

Testing. On the other hand, the research that has been done on the value of testing as a pedagogical procedure suggests that future research could have a valuable payoff. Frequent testing seems to be advantageous to the student. Optimal parameters (frequency of testing, length of test, difficulty of test) need to be determined for different grade levels and different kinds of students.

Tutoring. A sampling of the research done on tutoring reveals still another kind of result. Most of the research on tutoring has focused on tutoring in reading. However, a moderate number of studies in arithmetic suggests that well-trained tutors can have a positive effect. Unfortunately, the tutoring, and the testing of the effects, seems to have been confined to rote computation. It would be valuable to learn whether tutoring can have any effect on mathematics achievement at higher cognitive levels.

This list could be lengthened. However it does suggest that there are substantial opportunities for fruitful research in this general area, but that not all subtopics in this area are equally worthy of research support.

5. Students. The variables under this heading are similar to those discussed under the curriculum heading. There are some important questions which the evidence suggests can be answered. There are some variables which evidence indicates are irrelevant. There are some questions about which we know very little but which must be pursued. In particular, we know very little about the differences in the ways the various ethnic groups learn mathematics. Until we know much more about this, we are not going to be able to improve the teaching of basic skills to minority students.

6. The environment for learning. Probably enough is known about such matters as ability grouping (which affects the students' intellectual environment), class size, the physical environment in the classroom, etc., to be able to provide useful advice to classroom teachers and school administrators. However, this information is widely scattered and not easily available to these individuals. It would be worthwhile to collect, organize, and summarize this information in a form that school people would find helpful.

Summary

What I am suggesting is a large coordinated effort to collect, organize, disseminate, and extend what we now know about teaching and learning basic mathematical skills. A vast amount of information is scattered among journal articles, research reports, and dissertations and needs to be pulled together and organized. In some cases we will find that we already have enough information. In others we will find that the topic being investigated is irrelevant. In still other cases, it will be clear that further information is needed, and that further research is justified.

BASIC MATHEMATICAL SKILLS AND LEARNING

Nicholas A. Branca

What Are Basic Mathematical Skills and Learning?

This question can be interpreted in many ways. Various individuals and groups, including many school districts, have attempted to answer similar questions, and there now exist a number of documents that list mathematical skills and/or competencies and objectives and/or goals considered basic to the education of individuals. The more recent examples include Bell (1974), Edwards et al. (1972), National Assessment (1970), and School Mathematics Study Group (1972). My purpose is not to prepare another such list nor to compare and contrast the lists that already exist. Rather, I intend to give my interpretation of basic mathematical skills and learning and describe briefly how my conceptions relate to some of the documents cited.

In analyzing various statements regarding the objectives of mathematics education as part of an SMSG seminar (SMSG, 1972), objectives were found to vary over a limited number of dimensions including content, cognitive level, effective level, verifiability, feasibility, population, and form. Along with these dimensions, most objectives can be seen to fall into one of two major emphases. One emphasis deals with the utility of mathematics by individuals and society; the other deals with an understanding and appreciation of the nature and significance of mathematics as an abstract system.

Examples of objectives that can be classified under the first emphasis include the following:

1. Mastery of the four fundamental operations with whole numbers and fractions, written in decimal notation and in the common notation used for fractions (Commission on Mathematics, 1959, p. 19).

2. Ability to solve linear equations containing common or decimal fractions (Smith and Reeves, 1927, p. 202).

3. Given a whole number between 1 and 100, ability to identify it as a prime or composite number (Instructional Objectives Exchange, 1968, p. 58).

Examples of objectives that fall under the second emphasis include the following:

1. Appreciation of mathematical structure ("patterns") for example, properties of natural, rational, real, and complex numbers (Commission on Mathematics, 1959, p. iii).

2. Understanding that geometry is an ideal model of certain aspects of the physical world; a full appreciation of this is quite difficult (SMSG, 1972, p. 23).

3. Recognition of the evolutionary development of mathematics by noting the historical milestones in the development of mathematical ideas (Edwards et al., 1972, p. 676).

At various times in the history of mathematics education, one or the other emphasis has taken precedence. There have been advocates of the view that teaching objectives that pertain to the social utility of mathematics is the most important function of the schools. In some instances, the underlying structure of mathematics was not considered important and a stimulus-response treatment of mathematical objectives, in keeping with the psychological theories of Thorndike, was employed. With such an emphasis, many current lists of behavioral objectives that stress, for the most part, low level cognitive skills could be effectively utilized. In contrast to this, other advocates argued that teaching objectives that pertain to an understanding of the meaning of mathematics -- its nature, development, and appreciation -- is the more important function. In some instances, the mastery of basic computational skills and the applicability of mathematics to real world problems were ignored or rejected.

In my opinion, the notion of basic mathematical skills and learning is not adequately reflected in either of these two emphases. Rather, it consists of abilities (1) to integrate objectives both within and between the learning of mathematics for its usefulness and applicability and as a system in itself and (2) to understand how each of these two emphases supports and extends the other.

The [illegible]ing example may help clarify my conception of basic mathematical skills and learning. Many consider the realm of addition to include objectives such as learning the basic addition facts or how to add any two two-digit numbers with or without regrouping, as well as objectives such as learning the nature of binary operations and operational systems and their various mathematical properties. But, I do not think these are the basic mathematical skills and learning. Rather, basic skills and learning consist of the ability to relate these various objectives, to understand that the addition of basic facts, as well as two-digit, three-digit, ..., n-digit numbers, fractions, decimals, etc., all share common properties. Also, related systems of mathematics exist that possess these properties. Furthermore, there are many different uses of the systems and the various representations ranging from the simplest uses of addition for counting or totaling the number of elements in the union of sets to more sophisticated uses of addition properties in solving complex equations.

In relation to many of the lists of skills and/or competencies and objectives and/or goals that exist, my conception of basic mathematical skills and learning goes beyond a simple listing of behavioral statements, but relies on such statements for its substance. I conceive skills and learning as abilities that heavily rely on being able to organize data, search for relationships and patterns, generalize, and transfer. These skills and learning make mathematics meaningful as a tool and as a discipline.

What Are the Major Problems Related to Children's Acquisition of Basic Mathematical Skills and Learning, and What Role Should NIE Play in Addressing These Problems?

Before I can answer these questions, a more definitive statement of my conception of basic mathematical skills and learning must be developed and the relationship to the entire school curriculum, not just to the small percentage of time alloted to arithmetic, must be investigated. The conception now needs to be revised and refined, and the problem of measuring the attainment of basic mathematical skills and learning must be addressed. The question of whether all students can acquire these skills and learning must also be raised. I think teaching students only low level cognitive skills such as how to perform certain operations without the ability to relate, generalize, and transfer what is learned is not teaching them basic mathematical skills and learning. Perhaps certain individuals can learn only the how and/or would profit more from so doing. If so, this would create problems of identification.

The question of our expectations of students must also be addressed. Should we expect the same degree of learning from all students? I think not. Do we expect too much from some students too soon? I think so.

An additional problem related to students' acquisition deals with our scant knowledge of what fosters learning the abilities to organize information, recognize patterns, and generalize, and to transfer what is learned to new and different situations. Because they are at the heart of my conception of basic mathematical skills and learning, these abilities must receive our future attention.

A current major problem related to students' acquisition of mathematical skills is that of the mathematical competence of many elementary school teachers. There is much evidence indicating that elementary school teachers are not well enough prepared to teach all the mathematics that is expected of them. Also most teachers have not been taught basic mathematical skills and learning, as I conceptualize them, and, therefore, could not be expected to teach them to their students. If teachers are to be expected to teach these basic mathematical skills and learning, then the nature of their education in both mathematics and education must be changed.

The questions that have been raised give some direction to the second part of the original question dealing with the role of NIE in addressing the concept of basic mathematical skills and learning. In considering this role, support should be directed to four major areas. The first deals with a clarification of the nature of basic mathematical skills and learning and the problems associated with the measurement of these skills and learning. The second deals with basic questions of learning. The third deals with the relationship of basic mathematical skills and learning to the school's curriculum in general and with curriculum development. And the fourth deals with teacher education and includes both curriculum development and pedagogy.

The four areas are all related, and each requires the combined efforts of many individuals. To answer the various questions posed and to get maximum payoff among areas from the answers, I would propose that a central coordinating body be established to facilitate work in and communication among each of the four areas. The coordinating body should also be responsible for setting priorities and evaluating the work in each of the areas.

Looking ahead, I envision the following possible outcomes:

1. Some common consensus on what basic mathematical skills and learning are, differentiated for different individuals.

2. A system for measuring and evaluating these skills.

3. A growing body of basic research consisting of coordinated and integrated studies dealing with questions of learning basic mathematical skills and learning.

4. Integrated school curriculums that have been developed to teach these skills and learning based on research findings and practice.

5. And finally, successful teacher education programs that teach both the mathematical skills and learning and the spirit in which they are to be taught.

The key notion expressed is that of coordination and cooperation of the different areas concerned. Work can proceed in each area by different people at different locations, times, and rates. However, there will be some degree of dependence of areas on each other's findings, results, or products. Long range implementation procedures must also be coordinated.

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BASIC SKILLS AND LEARNING IN MATHEMATICS

Peter Braunfeld

The present paper will not answer the important, but difficult, questions suggested by the title: (1) What are the basic skills and learning in mathematics? (2) How can we improve the teaching of these basic skills and learning? Rather, it will attempt to sort out some of the issues these questions raise, and provide some framework in which we can start to think about these issues.

To begin, we might note that the word "basic" is probably not a property of something, but expresses a relation between two things, i.e., " x is basic to y."[1] To be sure, one frequently hears, "x is basic." However, if this is said by a careful speaker, then either the "to y" is clear from the context, or there is an enormous range of y's.

For example, we all agree that reading is a basic skill, because it is immediately clear to us that an enormous range of rather obvious activities are closed to an illiterate. Roughly speaking, x is basic to y if (1) x precedes y; and (2) if y is impossible (or at least very awkward) to do or understand without x.

Why Is Mathematics a Basic Skill?

Thus, in trying to sort out what are, or should be, the basic skills and learning in mathematics, we must first be as clear as possible about the questions: basic to what end, and for whom? This suggests discussing first why mathematics is, or ought to be, taught at all. The reasons that have been given historically fall into roughly four categories:

1. Mathematics is a tool for everyday life.

2. Mathematics is preparation for a variety of future careers.

3. Mathematics is a vehicle for generating and exercising critical thinking and problem-solving abilities.

4. Mathematics is done for its own sake as a stimulating, rewarding activity in itself.

I believe all four of these reasons for teaching mathematics are valid. None is spurious, and all should enter into the creation of a "good" mathematics curriculum for the schools. I also suspect that few will quarrel with my thesis in the abstract. The philosophical quarrels seem to center mainly about the degree of importance to be given to these various aspects of mathematics.

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[1] The use of the word "basic" seems to me to be similar to that of the word "relevant." That is, despite modern parlance, it simply doesn't make sense to ask whether x is relevant, but only if x is relevant to y.

Clearly the foregoing distinctions are only for analysis and discussion. It makes no sense whatever to insist that the introduction of any given topic into the mathematics curriculum be justified by reference to one, and only one, of the four aspects of mathematics. Indeed, ideally several, if not all, the four points above should enter into a good mathematics lesson: It is a wonderful thing when a topic is simultaneously useful in our everyday lives, is good preparation for a variety of popular careers, is intellectually stimulating, and is also esthetically attractive. Nevertheless, in this paper we shall consider each of the points in turn:

Mathematics in Everyday Life

It is beyond dispute that access to elementary arithmetic is extremely useful (i.e., basic) to the conduct of one's everyday affairs. By elementary arithmetic, I mean the four arithmetic operations with whole numbers and decimals, as well as a variety of elementary facts about percent and "simple" fractions.

The utility of arithmetic in everyday life presupposes at least two skills: (1) knowing which calculations are appropriate to solving the everyday problem at hand, and (2) having the means to carry out the appropriate calculations.

I have deliberately stated the situation in a precise, though somewhat awkward way. One thing is "perfectly clear": Without knowing "what to do" in a given situation, all the arithmetic skill in the world is totally useless. To know how, but not when, to add is precisely equivalent to knowing how to spell a word (e.g., "miscegenation") but not knowing what it means. We need to know how to spell a word if and only if we will use it in communicating our ideas. Similarly, we need to be able to make a calculation if and only if it is required to answer a question important to us.

Until recently, the student was forced to master the skills embraced under (2). However, with the invention and widespread distribution of electronic computers, cash registers, and especially hand calculators, "access" to arithmetic power no longer, in principle, presupposes that the student has mastered the clerical arithmetic algorithms.

It is hard to assess or predict the future impact of the machine calculators. On the one hand, it is clear to me that they should be introduced into schools and their use encouraged. On the other hand, I am convinced that they will never completely replace mental and pencil and paper calculations. To calculate 5 + 3 on a hand calculator seems to me somewhat equivalent to driving your Buick to the corner mailbox, or to using a typewriter for all of your written communications because you don't know how to write. This is not only a matter of being totally dependent on a mechanical device; rather, it is largely a matter of social convention. Society frequently expects, as a matter of custom and etiquette, that we can do things or know about things, even when those things are not, strictly speaking, really necessary nor useful. Knowing which fork to use at the dinner table contributes neither to good nutrition nor to the enjoyment of the meal, but we are nevertheless expected to know which is the "correct" fork.

Much of the current dispute between the public and the innovators in mathematics education appears to come from this distinction. The mathematics educators want the children to learn useful, enjoyable, sound mathematics; they can't understand why the parents are obsessed with mathematical "etiquette" (like knowing the arithmetic facts and algorithms "cold").

Imagine a wonderful new course on "food" in which the students are made aware of the principles of a balanced diet, vitamins, calories, etc. The curriculum developers are delighted that they have finally created a "real" course -- both intellectually stimulating and useful. They are dumbfounded when the public becomes irate because, after finishing this course, Johnny still eats his peas with his knife!

Perhaps I have overstated my case for the importance of social expectation versus genuine utility. Obviously, skills and learning that have outlived their usefulness will eventually wither away. But, during the period of transition, such as the one we are currently passing through, there sometimes arises violent, and even bitter controversy.

The mathematics education community must provide leadership during such a transition period, and explain to the public why emphasis is shifting from one set of topics and skills to another. (The "new math" has failed in this respect, to explain the shift to the public.) However, if the public still expects certain skills and learning in mathematics, then that expectation in itself makes those skills and learning "useful." Then, to spend some time and effort to teach these things in schools is not, it seems to me, pandering, but simply being realistic.

The arithmetic algorithms are mechanisms that enable us to crank out new numbers from old numbers. In principle, though not in practice, it matters little whether they are cranked out in your own mind, or whether you use an external, mechanized crank (i.e., a calculator). But, too frequently, the mills of arithmetic grind all too finely; we are not, or we should not be, interested in the exact answer, but only in an approximate answer. The reasons for this are, of course, well known: First, the data with which we calculate are usually only approximate, so that the precision we can obtain with exact calculations is spurious -- and, indeed, mischievous.

Second, large numbers are hard to think about. (That is, "large," not in magnitude but in the number of nonzero digits in its decimal expansion; e.g., 0.3547062 is "large" in this sense.) Formerly, one avoided large numbers because they were difficult to calculate by hand. Now, the calculators make that easy. But usually we need to think about the results of our calculations, and large numbers discourage thinking and understanding. For example, we often say that we need to get a "feel" for a number. Thus, the skills involved in estimating and approximating appear to be basic to general mathematics.

Another important reason for learning how to estimate and approximate is error detection. If anything, the calculators have increased, rather than decreased the importance of error detection. True, the calculators themselves rarely make mistakes; the trouble lies with the human input. It is, therefore, vital that we be able to take the output of the calculator and make a good judgment whether the answer is "reasonable."

It is interesting that the calculator can relieve us of the whole burden of making a lengthy, difficult calculation, but be of almost no help at all in judging whether the answer is reasonable. The reason for this is largely due to the nature of our system of positional notation. The utility (and glory) of this system is precisely that our modus operandi is almost entirely independent of the column we are working in, or the position of the decimal point. When calculating, an error in the ones' column is no different from an error in the millions' column. But, when we come to using the results of our calculation, these two errors have a vastly different significance.

Another basic skill for everyday life in connection with numbers seems to me to be the ability to present numerical data in a clear, concise manner and to interpret data. What I am thinking of might be subsumed under the general heading of qualitative statistics and rudimentary probability. We are, surrounded in our daily lives by tables, charts, and graphs of all kinds. Our ability to interpret these and to extract useful information from them seems to me important, indeed, basic. To be sure, many of us learn to do these things "on the street" so to speak, by osmosis, rather than formal training. Nevertheless, I believe the school mathematics class should provide additional opportunities for such activities. Similar remarks apply to the elementary ideas pertaining to probability and stochastic processes.

So far, I have talked only about the numerical component of the mathematics curriculum. Let me take up the situation vis-à-vis geometry. As for plane, intuitive geometry I find it hard to think of very much that is genuinely needed for the conduct of our everyday affairs. The little we ordinarily "need" can, it seems to me, be picked up with precious little formal schooling. Although we would all do well to spend more time on three-dimensional geometry -- representations of 3-D and the like -- I have no specific recommendations to make in that regard.

Mathematics in Careers

The utility of mathematics in a variety of socially useful and important vocations is often adduced as the most important reason for including it in the school curriculum. Whether or not this argument is valid, surely the historical reason for the extensive concern with teaching mathematics in school is precisely that it is needed for numerous professions and trades.

Our technological, industrial, business society is unquestionably based on the existence of a sizable corps of "technicians" who are more or less mathematically competent. Industry, business, and government are therefore directly concerned with maintaining a continuous flow of people who know, and can do, mathematics. This concern is evident in slogans such as the need to "maintain America's competitive position in the worlds of science, engineering, and business organization." It is also clear that insofar as the public deems our current technological society desirable and necessary, the public also has a direct interest in producing the needed population competent in mathematics.

There is a further argument that, because we live in a society based upon technology (and hence mathematics), it is important for everyone to understand mathematics. Generally, this argument has always left me cool. It is by no means clear to me that we cannot thoroughly enjoy the fruits of a tree that we do not understand. It seems to me that I live in many worlds that I do not understand. The labels on the food I eat assure me that it contains chemicals with names that are mysterious if not ominous; the economy that determines my daily standard of living is subject to forces I cannot comprehend; the safety of the automobile I drive is a subject of dispute between Mr. Nader and General Motors to which I can contribute little. I am involved in a long, tortuous litigation, where I would be completely lost without the continued and expensive advice of my attorney.

In view of all this, I find it difficult to believe that my ability to cope with and understand my world is greatly affected whether I can or cannot solve a quadratic equation, whether I do or do not know linear algebra, or whether I do or do not understand the fundamental theorem of calculus. I mean all this from a purely practical point of view. The question of whether such knowledge might have a cultural, civilizing, humanistic influence on me will be taken up in the next section.

It is perhaps interesting to comment briefly on why mathematics is so much more important in professional life than in everyday life. The major reason is that everyday life tends to be largely qualitative. The fine differences which the machinery of mathematics can crank out for me so well are often of no concern to me or, even if I could obtain them, the results are not worth the effort. Let us take two examples.

First, if a housewife has a recipe for 10 and will have 19 people for a dinner party, she would be compulsive indeed if she made any calculation beyond doubling the recipe. On the other hand, if a nurse is told to give a patient 190 percent of the standard dose of a powerful drug, she had better make the appropriate calculation. Mathematics beyond the rudimentary level has little to contribute to the world of dinner parties; it has much to tell us in the world of pharmaceuticals.

Second, suppose my home is x miles from store A and y miles from store B, and store A is selling a suit for $a and store B for $b. In this case, it would be foolish for me to calculate my gasoline expenses, the value of my driving time, the probabilities that I will prefer suit A to suit B, etc., etc., in order to arrive at a rational decision as to which store to visit first. Any saving I might obtain by one course of action versus the other is worth at most a rough intuitive guess. But, if store A and store B are owned by the same company, it may well be worth the company's time to devote considerable mathematical effort to deciding just where to locate a warehouse to supply these two stores.

The reasons for the difference in this case are not so much the nature and proficiency of the product -- as it was in the recipe and the drug prescription. It has to do with the "volume" under consideration, and shows us the immense practical significance of the Archimedean order principle! There is, I think, a pedagogical moral to this story. The child's world, I think, is on the whole even more qualitative than the typical adult's. The child lives in a world of colors, sounds, social interactions, qualities, not quantities. The child rarely deals in articles of such potency (e.g., the nurse's drug) or does such a "volume" of business that mathematics can contribute to his or her understandings of, or power over, "real" situations. This is why the overwhelming majority of story problems in textbooks are either contrived or irrelevant.

There are, of course, two fundamental difficulties in tying mathematics to career education. First, different vocations use very different parts of mathematics (e.g., the nuclear physicist, the small shopkeeper, the housebuilder, the physician, etc.). Second, we have no way of knowing what careers children will choose later in their lives. In view of this, we cannot do better than to give children a rather generalized mathematical background. But what does this mean? It seems to me that many, if not most, of the applications of mathematics arise in the context of a measurement of some quantity. For discrete quantities, a measurement will produce an integer; for continuous quantities a rational number -- hopefully, a decimal. However, for purposes of analysis and computation, it is extremely useful to extend the rational numbers to the real numbers. Thus, in some sense, I think the real numbers are what "it is all about."

A thorough familiarity with the field of real numbers still seems to me to be the best base we can provide for people who may be called upon to use mathemathics in their vocations. Thus, the content and aim of the traditional K through 12 curriculums -- or, for that matter, modern curriculums like SMSG -- were generally in the right direction. This is not to say that I agree with the details of the emphasis and modes of presentation in these curriculums (e.g., the emphasis on manipulation without any understanding in some traditional curriculums, or the absurd emphasis on logic and foundations in some of the modern curriculums).

In addition, I would like to push hard on the role of functions. Both within mathematics and in its application we are generally concerned with how y relates to x, and in many cases it turns out happily that y is indeed a function of x. The ability to look at the world from a function point of view seems to me extremely important. Generally, both the traditional and modern curriculums have not given functions the emphasis they deserve, and they have introduced them far too late. One exception is the CSMP elementary program, which does a beautiful job of introducing and developing the idea of function from the very beginning.

I have already mentioned the use of qualitative statistics and rudimentary probability in everyday life. The use of statistics and probability is of fundamental importance in many professions. Indeed, statistics seems to be the only contact with mathematics that many of my university colleagues in disciplines like English, biology, and history have. Hence, I would think that continued development of probability and statistical inference is basic for a careers-oriented math curriculum.

Finally, computers and data processors are playing an ever-increasing role in science, technology, and business. Thus, it seems obvious that children must be taught to appreciate what computers can and cannot do well and how to get computers to do the things they can do well.

Mathematics in Reasoning and Problem Solving

I have deliberately made a distinction here between "reasoning" and "problem solving." I use "reasoning" in the rather strict sense of being able to distinguish a correct from an incorrect argument. By "problem solving," I mean the much more general skills of how one can come to grips with a question, and ultimately, how one can pose the right questions.

Mathematics (especially geometry) has been traditionally held up as the vehicle for teaching and exercising the "art of reasoning." On the one hand, mathematics is simple enough that demonstrations (i.e., proofs) are possible, and on the other hand it is rich enough that the demonstrations can lead to interesting, nonobvious results. Traditionally, the skills of reasoning were taught largely by example (i.e., the teacher gave a demonstration at the board) and practice (i.e., the students did their own demonstration at home). The results of this method failed to satisfy many educators. The idea of proof was notoriously hard to teach; also, every mathematics educator had his or her own private collection of student "proofs" available for the amusement of colleagues.

To remedy this situation, some curriculum developers went the route of formalizing the idea of proof (i.e., reducing mathematics to formal logic). Now the correctness of a proof could be checked mechanically -- it no longer required judgment on the student's part. The efficacy of this method of teaching proof has been endlessly debated, and I do not know all the facts -- if anyone really does. My inclination is to believe that the percentage improvement gained is negligible. Large numbers of children will fail to learn to distinguish a proof from a random assortment of sentences. I suspect that the best we can do is a combination of the two methods: Lots of examples, lots of practice, a codification of some (but not all) the logical principles involved, and a willingness to admit that proving things in mathematics is simply not everyone's cup of tea.

There has also been endless debate on the question of "transfer." Can the skills of argument and critical reasoning developed in mathematics be applied by students to other disciplines and other areas of their lives? This is in some sense an empirical question that I am not competent to discuss, but I cannot believe that it really matters all that much.

Surely the ills of the world would not be significantly ameliorated if everyone really did modus ponens, and no one ever made the error of concluding $q \Rightarrow p$ from $p \Rightarrow q$. Of course, everyone has a favorite example of just such fallacious reasoning, culled from the daily paper, the advertising industry, or an unfortunate colleague. Such examples are carefully selected and, I think, self-serving. Moreover, the fact that they exist does not mean they are important. The world we live in requires good common sense, prudence, a sense of fairness, patience, compassion, and many other qualities of the mind and heart. The ability to stitch together a logically sound argument ranks low on my list.

Nevertheless, within mathematics itself the ability to follow an argument is surely basic. So, I would list the development of a sense of proof as one of the basic skills we are talking about, and I would think that a good curriculum would seize every opportunity from the beginning, to develop and refine this sense in the student.

The situation with general problem-solving skills is even more complex than that with reasoning. That they are basic to doing mathematics is beyond dispute. But, to what extent can such skills be taught? I think no one really knows. Carefully chosen examples, lots of practice, and some discussion of various modes of attack on a problem are undoubtedly useful in developing these skills. Consequently, we should include open-ended problems in the curriculum but not to the exclusion of some of the more routine matters that are also important.

Mathematics for Its Own Sake

There are those of us who like mathematics, and who enjoy studying it and doing it, even when it does not apply to our daily lives, our vocations, or our other intellectual interests and pursuits. Our situation is analogous to those who like to paint or play the piano, or do crossword puzzles or play tennis.

We have a perfect right to our love affair with mathematics, and as mathematics educators we have an obligation to open wide the door to those children who may come to feel as we do about mathematics. But, we have no right to *insist* that everyone *must* like mathematics, and, by the same token, we must not feel that *we* have failed if a substantial fraction of our students fail to be "turned on" as we would like.

As teachers of mathematics we are somewhat in the position of the old-fashioned marriage brokers: We certainly owe it to our young lady client to present her to the prospective groom in the best possible light. But there is a line where good salesmanship becomes fraud. To urge the lady to wear contact lenses instead of unbecoming spectacles is the former; to advertise that she comes with a substantial dowry when in fact she is penniless is the latter. Mathematics teachers have often been guilty of both bad salesmanship *and* fraud. Too often, the curriculum and the teacher have made mathematics such a dreary and dismal discipline that one wonders how any student survived 12 years. On the other hand, some modern innovators have so overemphasized the fun and games aspect of mathematics that the student has no idea of what mathematics is really all about.

Let me give another analogy: In reforming the English curriculum, it is legitimate to replace a dreary book like *Silas Marner* by a more modern "relevant" book like Hemingway's *Old Man and the Sea*. (There will, of course, be some children who won't like either choice!) But, it is *not* legitimate to replace *Macbeth* with Captain Marvel Comics because "the children enjoy that more."

Students have a right to be taught honest mathematics that is likely to be enjoyable. Thus, honesty and enjoyability are *basic* components of a good program. They are not skills to be acquired, or learning to be mastered, but they are basic, nevertheless.

Conclusions

For a variety of reasons, elementary arithmetic is still basic to everyday life. So is some ability to display, interpret, and process numerical data. There is little else that I can think of as *really basic*, but surely some intuitive geometry and probability are desirable.

Innumerable vocations above that of common laborer require more or less competence in a wide variety of mathematical disciplines. Obviously, we cannot prepare everyone optionally for an unknown career. The basics here would seem to be: a more than nodding acquaintance with the real numbers and with functions of a real variable, including most of the usual algebra of real numbers, algebraic functions, trigonometric functions, exponential, and logarithmic functions. The content of the traditional or SMSG-type programs suits me just fine. I would also include work on computer-related mathematics, such as programing, error detection, numerical analysis, and data processing techniques. As the students get older, and as their career choices and intellectual preferences become clearer, I would also hope to make calculus and linear algebra available to those who want it.

Viewing mathematics as a part of the entire spectrum of intellectual disciplines, I would certainly stress two things: First, mathematics deals with abstractions; i.e., ideas, not physical things. Second, we can reason about these things to produce new and interesting theorems. Finally, I would bend every effort to make the mathematics in the curriculum something enjoyable and attractive.

It is my experience that at conferences like this, optimism tends to run rampant, and wild claims are made for various new, innovative reforms. In a way, this is only natural: the work of the educational reformer is hard, filled with setbacks and disappointments. The educator needs to feel that really wonderful things will come from his or her work, or he or she would no doubt quit, and do something else. But overpromising in our business has, I think, already caused untold mischief; it has lost us much of our credibility with the public.

In the past 20 years, wave after wave of innovation and reform have swept over us. Each was "sold," not as a reasonable experiment or a likely step forward or a possible improvement. We were told they were panaceas for all our educational ills. But none were really the philosopher's stone. The reason, I suspect, is that the philosopher's stone simply doesn't exist. Recall the following: logic and set theory; discovery approach to learning; television; manipulatives; programed instruction; modules; "open" classrooms; behavioral goals; individualized learning packets; turtles; academic year institutes for teachers. The list can go on and on and on.

In closing, let me list just a few of the theses that have been floating around in our profession, that I now firmly believe to be false:

1. The Cambridge Report paints a realistic picture of children's interest in, and ability for, mathematics.

2. Most children and adults dislike mathematics and are bad at it almost wholly because they were badly taught.

3. The quality of most people's lives would be significantly improved if they were better at mathematics.

4. There exists a technological device (either currently available or yet to be invented) that will make nontrivial mathematics easily accessible to the vast majority of the population.

5. There exists a curriculum (either currently available or yet to be invented) that will make nontrivial mathematics easily available to the vast majority of the population.

6. If only the right point of leverage were found, competent mathematics teachers could be trained quickly and cheaply.

7. The application of principles of learning theory can contribute significantly to improving teaching.

This is not to say that improvements in our current situation are not possible and desirable. But, I fear they will be modest and expensive. I am sorry to end on such a negative note, but that is how I feel.

SOME NOTES ON BASIC MATHEMATICAL SKILLS AND LEARNING

Aaron D. Buchanan

What Are Basic Mathematical Skills and Learning?

Identifying basic mathematical skills, at least as they concern the formal education of children, is less a matter of proclaiming new truths than of rearticulating old goals and commonly shared social beliefs. This identification is less a matter of interpreting educational research than of reexamining social priorities. Basic skills in this context are less likely to emerge from constructs of how mathematicians do their thing than from an examination of the responsibilities of institutions for formal schooling. The issue is how these institutions can be more successful in discharging their responsibilities for the education of young children, or at least in identifying specific ways that they can avoid failure.

In areas of education related to mathematics, the first priority is development of quantitative literacy among learners. There can be little question that educational systems which do not give primary consideration to development of a basic core of quantitative skills will fail. The ability to quantify dimensions of real experience and to communicate these quantities is equivalent to the ability to read, and is second in importance only to the development of a language.

The most basic educational skills involve information processing and communication. The area of mathematical learning holds no exceptions. Children need to learn how to express quantities in given information-bearing contexts and to translate quantities which are already encoded. The ability to express "How Many?" "Which is more?" and "How much more?" are basic to any system for teaching mathematical skills which are fundamental to human interaction needs.

Obviously, the first skill of any consequence that children must master is the ability to count. Whether they learn to count by rote, by recognizing patterns in the system for ordering names for whole numbers, or by developing a basic understanding of what a numeration system is, is a side issue. No system can afford to fail in developing this skill at a rather early point in the child's learning experience.

Beyond the ability to express absolute quantities and compare them, there are basic skills for expressing relative quantities (one quantity in relationship to another). This ability is really little more than an extension of skills for expressing and comparing absolute quantities to more sophisticated expressions using fractions. While the use of fractions to express quantity clearly subordinates skills involving whole numbers in importance, it is not a culturally negotiable skill.

Additional basic skills in the quantitative core involve combining or separating quantities to form new ones in response to information needs. With small whole numbers, these skills are fairly intuitive and easy to learn. The difficulty begins to occur when it becomes desirable to make the process of identifying results of combinations and separations more efficient, and to extend this ability to larger and larger numbers.

It is at this point that pupils need to obtain rapid recall of combination and separation facts, and the procedures for using these facts to generate results when combining and separating larger numbers. It is also at this point that concern over what skills people need, and what skills a society expects schools to develop, begin to diverge.

Place value algorithms for independently computing the results of combinations and separations of quantities are probably not basic skills, certainly not in the same sense as counting to determine absolute quantities and expressing relationships between different quantities in an information context. A surfeit of devices is available on a mass scale for doing computations, and yet a system which does not produce pupils who can independently add, subtract, multiply, and, to some extent, divide whole numbers will probably come under public scrutiny as having failed. Skills in rounding and estimation, which are probably more important abilities than full-blown computation algorithms per se, do not have the same cultural price tag.

If any of the four major algorithms for computation is likely to be obviated in the near future, it is the algorithm for division, particularly division involving divisors of more than a single digit. The fact remains that expectation is wide for development of computation algorithms at reasonably specific points in the K-6 instructional sequence. Existing programs reflect this expectation. It is ironic that, with respect to computation algorithms, programs developed during the past 15 years were not that different from their predecessors; the development of the algorithms still required 5 years and 30-40 percent of the instructional sequence.

The remaining skill which is basic to human interaction needs is the application of numbers to measurement systems. Beginning with money and time and extending to length, weight, volume, and temperature, the ability to identify values in terms of standard units is a vital skill. Except for money, time, and length (to some extent), educational systems are not held rigidly accountable for development of measurement skills. Existing programs reflect these conditions.

Finally, geometry, while it occupies a significant part of instruction in existing programs, is probably not basic to minimal human needs. Certainly, there is no great social pressure to develop anything more than recognition of the most rudimentary geometric shapes among elementary school children. Existing programs reflect this in the amount of time spent reteaching the same concepts at successive grade levels.

Other areas of the elementary mathematics curriculum such as logic are bonus skills. In the case of logic, the informal skills which people really need to know are better taught as part of reading comprehension. Metric geometry is more important, but the parts of metric geometry which satisfy most human needs are more closely related to measurement systems than geometry per se.

What Are the Major Problems Related to Children's Acquisition of Basic Mathematical Skills and Learning?

Problems involved in learning basic skills are as much a part of existing programs for mathematical learning as they are a part of what has been learned about educational psychology. These problems are many and varied, and only a few of the most major ones are listed here.

How to teach basic skills. There is still very little hard evidence about the best way to teach mathematical skills. Development of basic skills in mathematics is not significantly better or worse today than 15 years ago. If contemporary mathematics instruction is to be faulted, it will be largely because too much attention is given to too small details, in an effort to force instruction to conform to modern mathematical constructs.

The result is often a confusing array of instructional prescriptions; "Don't say 'two-digit numbers,'" "Do say 'two-digit addends,'" "Don't refer to repeated addition as a model for multiplication," "Do refer to repeated subtraction as a model for division." In short, many programs have become victims of their own sophistication, and have frequently been forced to resort to arm waving to allude to the obvious.

How to teach computation algorithms. The elementary curriculum is hamstrung by the sequence for teaching computation algorithms. Until workable procedures are found for shunting large amounts of the detail in this sequence, there is really little room for maneuvering to teach anything else. Pupils cannot process information which requires combinations of large quantities until they have mastered the algorithm for computation, and each algorithm takes 2 to 3 years for development in the present form of instruction.

How to teach concepts. There is considerable desirability for exploiting children's natural experience with quantities to teach mathematical concepts, in much the same way that reading instruction builds from an existing vocabulary. The difficulty is that young children typically do not have a very large repertoire of number experiences to call upon. Their experience with absolute quantities is limited primarily to numbers less than 50, or even 100, which is only minimally acceptable as a context for teaching addition and subtraction algorithms, and entirely unacceptable for teaching multiplication and division algorithms.

The alternatives are either to restrict information processing to small numbers, or to systematically increase the scope of children's experience with larger quantities at the earliest points in the curriculum.

How to apply early learning. Instruction has tended to rely too much on the transfer mechanism in early mathematical learning. The strategy is to teach number and operation concepts and computation procedures in relative isolation, and then apply them to problem solving at each plateau in the sequence. This practice runs counter to many of the accumulated findings of psychological and educational research which show direct practice to be a more reliable (if less interesting) phenomenon.

How to establish sequence of problems. Logical analyses of sequences of mathematical problems have resulted in widespread belief (probably with some degree of error) that a skill which is logically subordinate to another skill constitutes an instructional prerequisite. The fact that pupils often have difficulty discriminating number properties from other properties should not be taken as a prescription for using mastery of constructs, such as conservation of number, as a readiness criterion for beginning instruction in identifying the number of a set. What it does imply is something about the breadth of experience that will be required before quantitative skills will reach a desired level of stability.

In the same spirit, there is no reason to believe that pupils must begin computation algorithms with addition of two-digit numbers. Instruction might very well begin with addition of three-digit numbers or even subtraction, albeit it's obvious that children who can't add two-digit numbers can't add numbers with three digits either.

How to improve flexibility. The inflexibility of the instructional sequence for most of mathematics leads to an inordinately high probability of repeated failure for pupils who do not develop a particular skill midway through the sequence. There should be instructional mechanisms which allow pupils who have not developed a computation algorithm to still participate satisfactorily in solution of verbal problems which involve the mathematical operation for which the algorithm was taught.

How to construct curriculums. The concept of spiral curriculum has been misapplied in many mathematics instruction programs. The result is often a rather artificial attenuation in skill development which will be continued in a later chapter or a later year of instruction. The practice of instructional recycling makes it difficult to identify points in the sequence where skill mastery should be expected, and to identify skills which are particularly important because of their instrumentality in development of later skills.

How to develop programs. Program development in mathematics resembles too closely the manufacture of consumer goods. Innovative programs are seldom developed because there is no market, and there is no market because such programs really don't exist, at least not in a comprehensive K-6 design.

How to use manipulative aids. Whatever the merits of manipulative aids to learning in mathematics, they typically do not adapt well to illustrations in textbooks.

What Role Should NIE Play in Addressing These Problems?

Given the curent state of mathematics instruction, NIE can participate significantly in several areas.

Basic research. Research into linguistic conventions for encoding quantitative information, and how children learn these conventions, is fundamental to effective programs in areas such as verbal problem solving. Present strategies for teaching pupils how to solve different kinds of verbal problems is unscientific at best, and a hodgepodge of arm waving at worst.

Some positive attention should be given to mitigating the effects of readiness skills and instructional prerequisites on subsequent learning. Instruction in mathematics, as it is currently represented in existing programs, is probably more linear than it needs to be. How much of this linearity is based on conventional ways of teaching mathematical skills rather than bona fide knowledge about skill development is difficult to tell. However, to confuse necessary skills with sufficient ones is a mistake that instructional science can ill afford. Furthermore, at least part of the research effort should be devoted to finding ways to circumvent many prerequisite skills, by artificial means if necessary, in order to remove constraints from the skill-development flow.

Program accountability. There is no acceptable mechanism that is widely available for examining and comparing program effects. Results of the National Assessment survey indicated poor performance by pupils and adults in solving consumer-oriented problems. However, there is no way to relate these results to specific programs and program comparisons.

Program implementation. No reliable or comprehensive studies of program implementation exist in education literature from widely available sources. It is difficult to know which programs are in use in different areas of the country, and how a given program is being used in conjunction with supplementary components and other programs.

Program development. The principal needs in this area are for alternative programs designed to teach basic skills. Since most programs are now based on instructional hypotheses that have little empirical verification, the least that can be done is to develop alternative hypotheses and apply them to a generation of instructional programs. Mathematics education cannot help but benefit from the availability of alternative program routes to development of basic skills.

BASIC MATHEMATICAL SKILLS AND LEARNING

Earl F. Colburn

What Are Basic Mathematical Skills and Learning?

Introduction. I am not a teacher, educator, or school administrator. I am a lawyer, an office lawyer. While that should entitle me to few credentials among math teachers, I will confess to a daughter who is a teacher, a son who is a math and chemistry major in college, and a wife who has a degree in astronomy! Identifying myself as "a lawyer" is really about as helpful as identifying any one of the conference participants as a "mathematician." In each case there are many colors and sizes. I would imagine that in each case, again, the identification must be made in terms of the educational experience and the intellectual discipline followed.

Similarity to the Law. With that declaration of my bias, a consideration of the similarities (or lack of them) between mathematics and the law, as disciplines, is legitimate -- maybe even essential. One obvious similarity is that both seek to derive a solution to a question or problem through a logical construct. But math, at least at first glance, applies concepts to obtain "the answer," while the law, by and large, applies concepts to obtain "an accommodation"; in this sense, law is more closely allied with the social sciences.

This apparent distinction may be illusory. Even as objective a discipline as mathematics actually works upon a rather formalized, and not necessarily provable, construct (set of rules), and when the construct is changed (say, as a consequence of Einsteinian theses), so might "the answer."

I understand that today different answers are in fact obtained depending on whether Euclidean or more modern geometry axioms are applied (such as relativity), and I am confident there are other theories I cannot even conceive of. Consequently, it may be inaccurate to speak of major differences between mathematics and law (and perhaps among any of the disciplines where the process of thinking is invoked).

Having rationalized this presentation, the "Basic Mathematical Skills and Learning" which I now suggest may indicate the reasonably close association of law with mathematics. Following NIE's lead that "skills" for this purpose includes "abilities, understandings, knowledge and so on," it seems to me that there are three major mathematics skills.

Major Mathematics Skills

1. The Rote Skill. The ability to absorb and "know" the arithmetic basics and tables (and perhaps even a modicum of algebra and geometry) has been the traditional aim of primary math classes. Today, there may be an attitude that this ability is no longer important, what with computers, pocket calculators,

specialization, and so on. Not so. If one of the goals of education is to prepare people for participation in an increasingly complex industrial society, this ability is as necessary as the ability to read and (maybe more than) the ability to write. Why? (1) The ability forms a base for greater learning capability. (2) Without some understanding of the relationship of numbers and their application in a given circumstance, socially acceptable (and economically rewarding) personal contribution may be impossible.

2. Application. The ability to apply the rote knowledge is the obverse side of the coin, as suggested above. It is not really a separate skill in the typical sense, but it is an essential ingredient in the equation. The ability to know when to use the raw material learned by rote is typically learned from experience, although there is perhaps some illustrative training in the classroom.

3. Understanding the Process. Rote learning and experiential knowledge of application provide the "survival" skills. However, the "cultural" or "societal" skill must be some understanding of mathematics as a system (or systems) of processes created to solve a problem or answer a question. At least at the primary levels of education, math is the only course where students are (or should be) forced to think (that is, apply by mental exercise, a rule to get from one point to another).

It is tempting to call this ability the exercise of logic. But that term suggests something pretty immutable and permanent, and neither law nor math can claim that. Math is undoubtedly more permanent (and less flexible!) than law, but has the ability to adapt and has been forced to do so. Both math and law are manmade rules to permit avoidance of, or solutions to, problems. Mathematics expresses and explains a system of relationships which are not self-determining. The law establishes a system of conduct to promote socially acceptable behavior in a system whose members have some self-determination.

Math and music are also frequently equated; I can see the "logic" of music more than the "logic" of math. In the abstract sense, law is not logical (although lawyers may be); math in the abstract may be logical (given the a priori construct), although mathematicians may not be. As a consequence, I prefer to think of this element of similarity in math and law as "analytical" rather than "logical."

At least at the primary levels, it appears that math has been taught and learned on a highly verbal basis in the United States. This verbal approach may actually tend to inhibit understanding of the process.

This hypothesis may illustrate a major distinction between the social sciences (including law) and the physical sciences (including mathematics). Even more, it may suggest a distinction between math and the other sciences. This could be true, because the other sciences generally involve something we can "see"; math does not.

Conclusion

I have tried to define the "basic mathematical skills and learning" in terms of the attributes which education attempts to develop in all students regardless of individual capability or capacity, in short, to meet the basic goal (expressed by most educational philosophers) of promoting intellectual self-sufficiency and freedom.

Perhaps the most important skill is the ability to understand (item 3), especially for the slow student, assuming that he is not actually mentally impaired. Some level of analytical thought can be attained by almost everyone. If the toddler accepts that all dogs bark and that Rover is a dog, the toddler can learn that Rover can bark.

What Are the Major Problems Related to Children's Acquisition of Basic Mathematical Skills and Learning, and What Role Should NIE Play in Addressing Them

With the personal background which I have already described, it is clear that I would be highly presumptuous to specify the current problems related to children's acquisition of basic math skills. My own experience is more than 40 years old; my vicarious experience (through my children) is exceedingly limited. So, I will identify no such problem, but I have some impressions and some comments.

Definition of Objectives

In almost any inquiry into a program or structure, it is helpful to define what we are trying to do. If my designation of basic math skills is accepted, it becomes fairly clear that -- in the math classroom -- we are trying to teach a math "language" (the rote material) and at the same time promote some kind of independent thinking. It is important that the student learn the fundamental rules, but it is also important that he knows that the rules are only a convenient framework from which to explore less clear concepts.

More Interesting Applications

In my experience, and I attended very good schools, math could be pretty tasteless gruel. It was almost all rote, with little application, and no understanding. I do not see why the application could not come earlier, and in a more stimulating fashion. The old grocery store examples are all right, but there are more everyday, interesting sources available, particularly in the vocational fields (such as automotive, carpentry, and plumbing). The coming of the metric system generally could be a boon to stimulate the young student, if properly presented.

Greater use in the classroom of representatives from the "real world" could evoke interest and promote more attention-getting stimuli for thought provocation. This program should not be too expensive and probably could be manned mostly by volunteers from private life. In providing the idea and offering a few specific suggestions for administering the program, NIE could contribute significantly. The program could be successful in the classroom, however, only if the teacher and the school administrator have enough creativity and adaptability to blend the outsider's presentation into the normal course work.

All of this leads me to the final concern. Do our math teachers realize that the theoretical thinking (analytical) element of math instruction is as critical as the rote? NIE might well focus on this question since it relates to the teaching of teachers.

I suggest that NIE (and especially the Basic Skills Group) formulate a policy that acknowledges the need to stimulate thought as well as to impart information, at the earliest possible age. This policy would promote the idea that mathematics is indeed a building block method of learning the analytical (maybe even "logical") approach; perhaps it would even be an admission that the "why" is more important than the "how to." This should go far to meet the educational goal of "intellectual self-sufficiency and freedom."

One last word in this vein. John Dewey emphasized the reciprocal learning between teacher and student. All good teachers know and practice this principle, consciously or not. Emphasis on the need to develop basic mathematical skills through such an interpersonal approach would probably reduce the dreariness of early instruction but, more important, prove to be a more rewarding and productive experience for both student and teacher.

BASIC SKILLS AND LEARNING IN MATHEMATICS

Robert B. Davis

What Are the Basic Mathematical Skills and Learning?

Basic Skills Defined

Before considering the following specific list, please notice that we are not construing "basic" to mean that these skills should necessarily be taught in the elementary schools. Nowadays nearly everyone goes well beyond grade 6, and some of the things that "everyone should know" are better learned by more mature students. Indeed, it is our guess that some of the weaknesses in present curriculums result from trying to teach certain topics too early -- the division of fractions would be a case in point.

While there is considerable evidence that virtually all children can learn more mathematics in elementary school than they presently do, there is a big difference between those specific mathematical tasks that are appropriate for young children, and those that are not.

Nonetheless, a very large portion of the "basic" material in the following list would presumably be learned in grades K through 6.

Basic Skills and Learning

"Basic skills and learning" certainly do include a full range of appropriate algorithms, including long division and the four operations on fractions. Some of the work on fractions might be deferred until later grades, and speed should be deemphasized both in teaching and testing.

"Basic skills and learning" must include the realization that arithmetic describes reality, and must therefore also include skill in translating (in both directions) between "reality" and "arithmetic statements"; e.g., $8 \div 2$ should be translatable to "real stories" such as "sharing 8 cookies equally between 2 people," or "having 8 firecrackers and setting off 2 each day, for how many days?"

The "basic" curriculum should include a few of the most powerful heuristics -- at least the following:

1. Solving and studying a similar easier problem.

2. Identifying what makes a problem difficult, and then trying to circumvent, eliminate, or otherwise deal with the specific complicating difficulty (as in eliminating the fractions in an equation by multiplying both sides by the denominators).

3. Breaking a problem up into appropriately chosen subproblems.

4. Making use of successive approximations.

5. "Doing what you can do."

This last heuristic is dangerous, because it can lead you astray, but it does seem to help some students get started, and "getting started" is often a big part of the struggle.

The "basic" list certainly includes a reasonable mastery of measurement, and probably also includes some mastery of estimation and of approximation.

The "basic" curriculum must go beyond the traditional rudiments of arithmetic to include something like the following list:

1. Functions.
    a. linear, nonlinear, exponential
    b. rate of growth

2. Interest, compound interest, some relevant economic facts and phenomena (e.g., money invested at 15 percent interest will double in 5 years).

3. Graphs, especially in relation to rate of growth.

4. Fundamental economic graphs, such as the graph on the inside back page of the Wall Street Journal.

5. Simple ideas of statistics, with a generous experience of practical uses to illuminate phenomena that are important in their own right.

The "basic" curriculum should also provide some sense of the size of numbers, and what this means, especially in relation to important current social issues:

1. Population growth.

2. Demographics (e.g., how many Americans are in the age range n years to n + 5 years, for n = 0 to n = 65?).

3. Distribution of incomes.

4. World food resources.

5. How much it costs to go to college.

6. Energy sources and energy consumption.

7. How much it costs to build a hospital, a highway interchange, a motel, a library, an apartment house, a bridge, or to buy a farm.

8. Annual and lifetime earnings of different occupations.

9. How many Americans, Chinese, Indians, Japanese, Russians there are.

"Basic skills and learning" must also include the development of effective personal habits and attitudes regarding mathematics and the performance of mathematical tasks:

1. Knowing what mathematics can and cannot do for you, e.g., a good mathematical analysis may pinpoint how some process -- making fudge, say -- is failing; mathematics cannot settle questions of values (although here, too, a good mathematical analysis may clarify the alternatives).

2. Realizing that very often mathematics can be "figured out" or "thought out" or "discovered" (or "invented"); it is not true that if no one has told you, you can't possibly figure out what to do.

3. Recognizing that sometimes mathematics cannot be "thought out" -- there are situations that must wait for special insights or special flashes of genius (or even a lucky guess), and they may have to wait a long time.

4. Realistically appraising your own ability to "think through" or "discover" mathematical ideas or techniques.

5. Developing a positive attitude toward what mathematics is capable of offering (as in helping us solve problems that might otherwise elude us, from the spontaneous combustion of a pile of coal to the death by freezing of infants in England).

6. Developing a positive attitude toward what you, the student, can do within mathematics.

7. Developing a positive attitude toward what you, yourself, can do in using mathematics (to make a shirt, to understand inflation).

8. Setting reasonably ambitious goals for yourself. (Most people underestimate their potential ability to do mathematics, often by orders of magnitude.)

9. Forming personal habits appropriate to the pursuit of these goals.

10. Understanding the power and the limitations of abstractions and idealizations.

11. Acquiring, to the fullest extent possible, the ability to learn mathematics from independent reading. (This may turn out to be too ambitious.)

The "basic" curriculum should also include the mastery of appropriate metalanguages to communicate about mathematics, and to communicate by using mathematics.

Who Shall Be Serfs?

Undoubtedly the preceding list will seem to many readers to be wildly unrealistic, to lose sight of any reasonable meaning of "basic." It represents my personal assessment, after some years of agonizing reappraisals, of the least that we can provide a child, if we want him or her to become a fully-participating citizen...and I greatly fear that we will not achieve this necessary minimum. We do not now provide adequate medical care for all of our citizens, nor adequate nutrition, nor adequate housing, nor adequate protection from crime, nor adequate legal assistance. Nor have we ever provided everyone with an adequate minimum education.

It is for this reason that some observers say that our schools are really filters, separating out those who shall become doctors, lawyers, scientists, and airline pilots, from those who will become dishwashers, waitresses, bellhops, gasoline station attendants, and garbage collectors.

Who will be the serfs? In the past they have been the children of the poor, blacks, ethnic minorities, and women. (Recent studies of women have revealed that women typically avoid fields that involve mathematics, and that there is a sizable positive correlation between how much math a field involves, and how well it pays. This, then, is one more way that women have come to find themselves in an inferior economic position.)

We have never done much to help the serfs, unless and until they became organized and compelled us to see their plight with new eyes.

The day is fast approaching when neither blacks, nor the poor, nor women, nor minorities can be wholesaled into serfdom. Who will the new serfs be?

I fear it may be those who have not developed effective learning and performance heuristics, or who have not developed appropriate habits and emotional patterns -- or, did not develop them early enough in life.

There, of course, is the rub! A child can easily fail to develop the cognitive skills or emotional patterns or personal habits that can lead adults to a nonserf life. Is this a "crime" for which that child should be sentenced to a lifetime of serflike employment? And if not, is it not an adult responsibility to see that every child -- or as near to that as we can get -- keeps his or her options open?

It will be all too easy to fill the ranks of future serfs with children who unwittingly "volunteer" before they have any realization of the choices that lie before them. Can we thus take advantage of the innocence and naiveté of youth, and still maintain our own clear conscience?

And what will happen when the indolents follow the path of the blacks, Spanish, Indians, women, and ethnic minorities, and demand equality? Will we create a school curriculum in which sloth is no disadvantage? I very much fear we will -- indeed, we are well on our way down that road already.

The future is not one of affluence -- it will call for ingenuity, competence, realism, effectiveness, and hard work. Our "basic" curriculum must be an education for fully-participating citizens, competent citizens...anything less is a program for shanghaiing serfs.

If all that we teach in the "basic" curriculum is skill in the long division algorithm, and a few similar things, the result will not be worth much. We shall have taught some humans to give a poor imitation of a hand-held calculator; that amounts to a program for creating serfs.

What Are the Major Problems Related to Children's Acquisition of Basic Mathematical Skills and Learning?

Range of Problems

One can list these problems under several headings: teachers, children, parents, schools, learning materials, learning experiences. My personal opinion is that we are dealing with a systems problem on quite a large scale. (Indeed, the failure of the electric power grid in the Northeast a few years ago could be the metaphor for our age. Nearly everything turns out, upon examination, to be hooked into a very large system, and when severe problems appear these systems are usually beyond our ability to control -- at least by any quick and easy method. The reader is invited to select his or her own favorite problems, from the delivery of quality medical services to the effective use of tax dollars, and not exluding prison reform or the foreign policy of the United States.)

While I am sure NIE must seek a practical agenda on which to begin work immediately, I myself -- suspecting a deeply interrelated systems problem -- cannot present a discussion that omits systems considerations, though I am fully aware of the risk that this can rob us of any will to take appropriate actions in areas where some progress may actually be possible.

Problems of Children

Focusing first on children, we can identify three kinds of problems:

1. Cognitive problems. These may be divided into two kinds: those relating to misconceptions, such as those studied by Erlwanger (1974), and those related to a child's progress in developing heuristics. (We will be publishing soon some data on the heuristics question, suggesting that getting students to think heuristically can be very difficult, if not impossible.)

Cognitive problems, of both types, strike me as central to the entire question of helping children learn mathematics. Our work on PLATO courseware (see, e.g., Dugdale and Kibbey, 1975) convinces us that one must think of the task in terms of the student's a priori cognitive structure, the interactions between the student's cognitive structure and the learning experience, and the resultant change in the student's cognitive structure, comparing his a posteriori concepts with those that you wish him to have. That is a regrettably wordy way to put it, but it attempts to present a point of view quite different from the more common ones. Education more commonly deals with the malleability of a child's external behavior: what we can get him to profess to, or to repeat from memory, with little concern for what he is thinking. But, in mathematics -- even "basic" mathematics -- thinking is the name of the game.

If a cognitive orientation comes to prevail, it will have important consequences for curriculum, pedagogy, and evaluation. Instead of broad, generally quantified "knowledge" -- such as saying that a child "performs mathematically at the fourth grade level," one would seek a precise description in terms of detailed "hurdles" or "milestones" in the building of cognitive structure, some of the kinds of things presented in Ginsburg (1975).

Thus, for example, for young children, one hurdle in learning to count is the task of realizing that the words "one," "two," "three," and so on are not names like "Joan" and "Sally." If you are "Joan" today, you cannot be "Sally" tomorrow, and young children attempt to use "one" and "two" in this same way.

Thus, one child mentioned in Ginsburg counted her sandwiches at teatime: "one, two, three." The governess said "Eat one." The child complied. The governess asked "how many do you have left?" The child replied "three." "But," objected the governess, "you ate one." "I know," said the child, "but I still have two and three left."

Evaluation based on specific hurdles or milestones would be entirely different, both in nature and in its impact on education, from the norm-referenced tests commonly used today.

2. Motivational problems. Space forbids dwelling on this here, but observing students in our lab school has convinced me that we need -- and could in fact create -- an entirely new theory of motivation, one that sees motivation as relative to alternatives (maybe I really want to finish this problem, until my brother reminds me that Star Trek is about to come on on TV), that recognizes several broad kinds of motivation (what my self-respect makes me do, what I do for direct intrinsic reasons, what I do instrumentally in order to achieve some other goal), that discriminates being motivated to solve this problem so as to avoid the teacher's censure tomorrow in class from being motivated to solve this problem to see if my method really works from being motivated to solve this probelm in the hope of understanding it better, and that focuses particularly on long term forms of motivations; e.g., working hard on math in order to understand it because I want to be an engineer.

3. "Pathology." We have been studying cases of children who have trouble learning. A number of interesting phenomena appear: for example, a boy, quite clever in math, whose progress is severely limited by his stubborn determination to figure everything out for himself. He refuses, for the most part, to learn from any teacher. His own methods are often good, but they are not perfect; if he were more willing to learn from others he could progress much faster.

Phenomena of this sort we have been labeling "pathology," not to exaggerate its severity, but to call attention to its existence, diversity, and importance.

Learning Experiences

Lacking a stronger and more detailed cognitive theory, we are not in a good position to analyze learning experiences, but major improvements in our ability to do precisely this are urgently needed. McKnight (1975) in a recent doctoral dissertation at the University of Illinois-Urbana addresses this problem.

Influence of Parents

During the 1974-75 academic year, Jody Douglas, an anthropologist, lived briefly in the homes of a number of seventh grade students. Her results (Douglas,

1975) are eye-openers. Speaking personally, I have never before seen any educational research that has made me reassess my own values (and my practices, both as a parent and as a teacher) to anything like the extent that the Douglas studies have. My children would, I am now convinced, be far better students if I had behaved differently as a parent. Further studies of this sort could have a very large payoff, to harvest some of the traditional wisdom that some parents call on in guiding the growth of their children. The Douglas studies leave no doubt that different patterns of parental behavior can make a tremendous difference, including a number of possibilities that few of us have ever even imagined.

Culture

There are many different cultures in the United States today, but the dominant culture does not serve us well in regard to the learning of mathematics.

1. There is a wrong idea of what mathematics is, and a wrong idea of how you learn it.

2. Specific expectations about careers, student performance possibilities, etc., are often harmful to students; as one example, consider the studies cited earlier about women's frequent avoidance of mathematics, and the consequent effect on their career choices and lifetime earnings.

3. Our culture has come to include some general attitudes toward work, discipline, excellence, objectivity, and competition that are not at all helpful im promoting the study of mathematics.

Little of this may have immediate relevance, except as phenomena that we need to understand better, but sooner or later there will be possibilities of attempting to shape these attitudes and values, quite possibly partly through television. And when that time comes, we will be much better off if we have already on hand some good studies of the effects that various themes in contemporary culture have on the success that children experience in learning mathematics.

Training in Universities

One university on which I have reliable data has found that about 25 percent of the undergraduates entering elementary education -- and thus, preparing to be elementary school teachers -- are unable to answer correctly many of the problems on standardized tests of elementary school arithmetic. They have trouble getting correct answers to problems in long division, fractions, decimals, and percent. No effective remediation is provided -- though the students suffer through a course in finite fields and introductory set notation -- and at graduation about the same percent of students still show about the same pattern of weakness in arithmetic. At this point they become elementary school teachers.

That this is very largely a systems problem is indicated by the fact that there are people who could create excellent courses that would be helpful to these elementary education majors -- and to all other prospective elementary school teachers. However, the division of responsibilities between the mathematics department and the college of education has thus far precluded any effective solution of the problem, given the committe structure, career possibilities, and internal economics within both units.

Learning Materials: The Effect of Format Uncertainties

At present a number of different settings and formats are employed in different classrooms. There is teacher-directed study of a whole class -- familiar to most people my age from our own school experience. There is a "discussion" and "sharing" format. There is a format in which the class is divided into several small cooperative groups. In another format, students work on rather extensive projects of their own design. There are several variants of "open education" and several variants of "individualized study."

Present classroom practice is in a state of flux -- few, if any, of these formats presently operate in a reasonably stable and well-defined form.

Yet each different format requires different kinds of learning materials in order to be optimally effective. Moreover, each is better able to handle certain aspects of the curriculum, and less suitable for some other aspects.

This situation militates against the production of effective learning materials, and must be added to the list of reasons why more children are not learning more mathematics.

Problems in Theory and Problems in Evaluation

These go hand in hand. To the extent that we are unsure of how a child's cognitive structure is built up, or why one child works hard and effectively in learning mathematics, while a second child works hard but cannot seem to untangle his or her own thoughts, and a third simply does not work at all -- to the extent that the details of these phenomena are mysteries to us, we cannot expect evaluation to give us much useful information. Because we do not know what detailed processes must be carried out, we have great difficulty in determining to what extent they have been carried out.

Significant progress in understanding these processes may be reasonably near at hand, especially if present efforts continue, as one hopes they will.

Schools

Mathematics education should also begin to take into account some of the recently created or presently planned smaller special purpose secondary schools, and in particular the small special purpose engineering high schools now being created in Newark, New Jersey, in Houston, Texas, and elsewhere.

For many young people the best answer to motivational problems seems to be to offer them a possibility of an early and clearcut career identification -- "I'm going to be an engineer," or "I'm going to be a doctor." It does not matter that careers may be changed later -- for example, one-half of the students receiving bachelor's degrees in engineering from Princeton go on to medical school! -- the point is that some clearcut career identification should be available at each stage, even though the young person may change this identification several times.

Too many schools offer no clearcut career identification at all. (Cf. the "Report of the Chatham Summer Study Group," of the Alfred P. Sloan Foundation, summer, 1975.)

Choice Mechanisms and Progress

It is hard to imagine *any* schemes for the improvement of education producing important results in the absence of some general driving force for quality and progress. At present, the only effective forces of this type are the consciences of individual teachers and administrators, and the expectations of students. Valuable as these often are, for a system of the size and complexity of schools this is not enough.

Major improvements in education demand something more. This demand is widely recognized today and finds expression in legislative forays into "accountability," into big city attempts at "decentralization" and "community control," and especially into voucher plans (cf., e.g., Friedman, 1972). Of all these, voucher plans seem to have the most promise, for reasons given by Friedman (1972). Without some general driving force toward improvement, no great progress in mathematics learning would seem likely to occur.

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REMARKS ON BASIC SKILLS AND LEARNING IN MATHEMATICS

James T. Fey

Review of Recent Efforts

The challenge to describe basic skills and learning in school mathematics is an assignment full of pitfalls. In the past 5 years, hundreds of mathematics educators, school systems, professional groups, and the National Assessment have been busily composing taxonomies of fundamental objectives for mathematics instruction at various grade levels. With few exceptions, these efforts to establish a reasonable list of basic skills have been failures. There has been no general agreement among the competing groups. Moreover, the implementation of the various lists as curriculum guidelines threatens to produce fragmented mathematics programs that resemble occupational training more than they resemble education in mathematical methods and understandings likely to be of long range value.

Reasons for Failure. There are many reasons for this failure to establish sound and widely accepted outlines of basic skills and learning.

First, in building skill lists, the most common reference points have been the mathematical demands of "daily life" -- and often yesterday's life! The weakness in this approach should be obvious in our future-shock world. Evidence that the lesson is not easily learned appears in the latest National Assessment report on consumer mathematics. The NAEP publicists highlighted two unit-pricing exercises and expressed shock that "less than one-half of the 17-year olds and young adults could successfully determine the most economical size of a product," ignoring the fact that unit-pricing skills tested in the exercises are of little use in contemporary or future stores.

A second major weakness of most "basic skill" lists is the tendency for most to resemble the menu at a Chinese restaurant. It takes a very knowledgeable teacher to build a coherent and balanced instructional repast out of the myriad details.

Third, the elements of basic skill lists tend to be specific items, not amenable to the kind of research that will have any broad applicability in teaching practice. Of course, these very specific skills tend also to stress low level cognitive abilities.

Problems of Definition. An even more fundamental problem in establishing definition of "basic skills and learning" is the immense variety of meaning attributed to the term "basic." If interpreted as the minimum needed to survive in society today, the list of skills should be extremely short. The skills required to be an astute consumer constitute a narrow, but more demanding, list. Moreover, the mathematical abilities required to be an effective citizen, able to comprehend our social and technological environment, constitute a curriculum far deeper and broader than most extant basic skill lists.

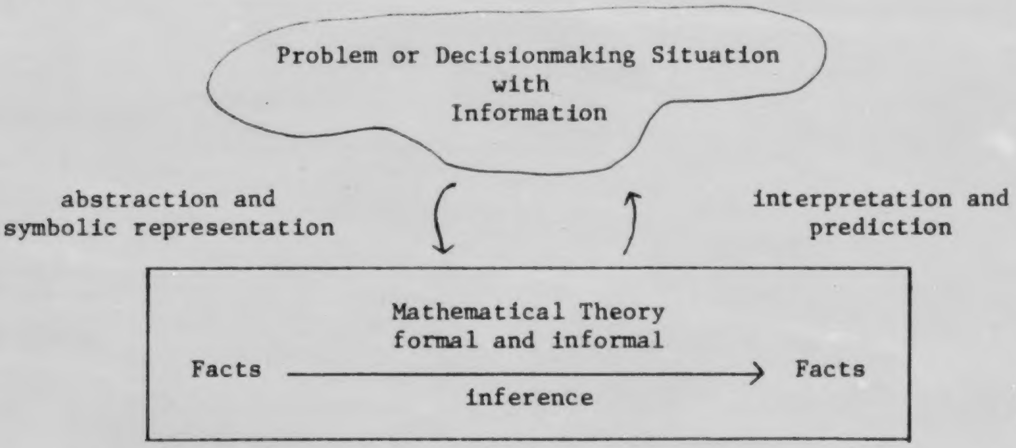
If one wants to be assured that students are acquiring all the mathematical abilities in grades K-12 that will keep open a wide range of later career options, basic skills probably include most of the content in current school programs, plus some ideas not currently stressed. To say that basic skill and learning encompasses all of the above definitions begs the question in a way that is of no value for outlining a coherent program of needed research and development in mathematics education.

The "survival" and "consumer" definitions of basic skill are so narrow and pessimistic about the goals and potential of school mathematics that neither the public nor the profession should devote serious attention to them as guides for school programs, even though many teachers and parents are currently clamoring for just such lists.

Focusing on the next level of "basic" mathematical abilities -- sufficient for effective citizenship and ability to comprehend the social and technological environment -- seems a more worthy aim and also yields useful insight into the current research and development needs in mathematics instruction.

Max Bell's paper, "What Does 'Everyman' Really Need from School Mathematics," makes an admirable start at describing the objectives of such a program. It is successful precisely because it does not take an extremely pessimistic view of what school mathematics should accomplish for "Everyman," nor by a "now"-oriented view of useful mathematics.

Devising a Curriculum. It is not plausible to try to make a curriculum simply by converting the Bell list into a sequence of behavioral objectives in one-to-one correspondence with instructional units. However, the program does come across as an uneven list without apparent structure. One way of fitting the "Everyman" abilities into a coherent outline of school mathematics is to use a schematic representation of mathematical activity such as the following:



This sketch is a diagram of the mathematical modeling process and is more simple and orderly than the actual activity. Thus labeled, it might appear to apply only to high level uses of mathematics -- at least the term "mathematical modeling" has that connotation. However, I think any situation in mathematics fits the model in some basic form.

A problem or decisionmaking situation involves information about attributes of, relations between, physical objects, quantities, perceptions, and so on. The problem is caused by missing information, or by the observer's inability to predict behavior of the situation. To help derive the needed information or predict the functioning of the problem system, one identifies the components of the problem setting with objects and relations of some structured mathematical theory, infers new fact and relations in the theory, and interprets these results in the context of the problem. To carry out the process successfully, one must know some mathematics, have the ability to recognize representations of that mathematics in problem situations, and the converse ability to interpret the findings of mathematical reasoning in real life situations.

Weaknesses in Applications. Traditional mathematics instruction has focused almost exclusively on activity in the "Mathematical Theory" box. The translation of verbal phrases into mathematical symbolism, that passes for applications in most school courses, is so limited and is taught in such a routine fashion, that it hardly merits the term "model building." However, recent technological developments with calculators and computers have so seriously weakened the case for manipulative facility as a basic in mathematics teaching, that the other phases of the problem-solving cycle can now be given a much more prominent role.

How does the preceding analysis lead to a definition of basic skills and learning? Where does the Max Bell list fit into the picture? In broad terms, basic mathematical competence consists of ability to perform each phase of the problem-solving decisionmaking process and the natural predisposition to use that process when it is appropriate.

Bell's List of Skills. The "Everyman" list is a first approximation to analysis of the constituent skills that play a role in the three phases of the broader process.

First, consider these categories presented by Bell, which all play a role in the representation of problem or decisionmaking situations:

1. The main uses of numbers.

6. Links between the "world of mathematics" and the "world of reality."

7. Uses of variables.

10. Descriptive statistics.

11. Geometric relations in plane and space (visual sensitivity).

Second, consider these categories which Bell describes:

2. Efficient and informed use of computational algorithms.

3. Relations such as equal, equivalent, less or greater, congruent, similar, parallel, perpendicular, subset, etc.

5. Confident, ready, and informed use of estimates and approximations.

8. Correspondences, mappings, function, transformations.

9. Basic logic.

11. Geometric relations in plane and space (standard geometry properties and their application.

And finally, let us look at the categories which Bell tells us play a role in the translation of mathematical information to real life situations.

10.1 Prediction of mass behavior vs. unpredictability of single events.

12. Interpretation of informational graphs.

13. Computer uses (limitations).

(The numbers correspond to those in Bell's 1974 article.)

Several other "Everyman" categories include subcategories that appear in all three phases of the total process. For instance, the description of fundamentals measurement concepts includes problems of units, instrumentation, and approximation that fit the model building phase; derived measures via formulas are used in the mathematical theory calculations; and then practical problems like approximation and error occur in the interpretation phase.

It is not hard to quibble with the specific descriptions of groupings of the "Everyman" list. What I am suggesting is a framework for building a more refined list of basic skills and learnings in the same spirit as that proposed by Bell. For generating important and needed research questions, the list is already very provocative.

Needed Research in Basic Skills and Learning

Substance and Strategy

Recommendations on research and development activity centered around basic skills and learning fall into two categories: substance and strategy. The model of basic skills suggested above calls on students to exhibit behavior of fundamental interest in psychology -- abstraction, generalization, use of symbol systems, pattern recognition, logical thought, memory, geometric perception, and so on.

Despite continuing interest in these areas, psychology today has little to offer mathematics education in the way of advice on how to design instruction to enhance students' skills with these processes as they occur in normal problem-solving situations. Major efforts in the near future must be more pragmatic. We must approach the problems of effective instruction in the (newly conceived) basic skills and learning from the point of view mathematics educators know best: design, delivery, and evaluation of instruction.

Mathematics education has focused attention predominantly on the development of skills within a well formed mathematical theory. But future efforts should be directed at better understanding of the mechanisms by which problems are formulated and results interpreted, bridging the gap between real life situations and mathematics, or between less completely understood aspects of a mathematical theory and the routine parts.

One useful first step would be to procure some detailed baseline information about the abilities of students and adults in the various component skills of the modeling and interpretation process. To get this kind of information we will need a whole new array of evaluation tasks and strategies. We must get information far different than "got it right" or "got it wrong." The situations must go far beyond the typical verbal formulations. For instance, a National Assessment item answered correctly by only 47 percent of 17-year olds asked:

> A parking lot charges 35 cents for the first hour and 25 cents for each additional hour or fraction of an hour. For a car parked from 10:45 in the morning until 3:05 in the afternoon, how much money should be charged?

Despite the fact that the problems were read aloud to all examinees, there is no guarantee that comprehension came with the oral presentation. The corresponding real life situation presents itself much differently than this sort of unattractive paragraph. And even allowing for NAEP style of reporting main types of errors, we don't learn much about how examinees responded to the situation and what led them astray.

Shift in Emphasis Needed. To some extent the needs in basic skill development are more a matter of redirecting curricular emphasis than of solving specific instructional problems. The school program currently devotes too little attention to many items I have labeled basic skills. One way to correct this imbalance is to support curriculum development activity by demonstrating instructional experiences that put a proper balance on the various components. In all likelihood this would not mean massive reconstruction of the school curriculum, but would require supplementary material and modules that would first enrich current programs and gradually make their way into standard practice.

Whatever curriculum development model is chosen, it is extremely important that broad dissemination and implementation plans be a central part of the process. We have too many examples of good programs not at all widely used because of weaknesses in the dissemination process.

My last point concerns strategy for work on improved basic skill programs in mathematics. First, it would be a serious mistake for the mathematical community or NIE to embark on a massive, widely publicized "right to math" type of program. Though the Euclid conference was apparently prompted by visions of an analogy to national programs in reading, this hope presents at least two fundamental difficulties.

First, goals like "mathematical literacy for every American by 1985" are inherently unrealizable (and the writers of such slogans know it) because they suggest a false analogy to medical problems like eliminating yellow fever. Education is a continuing process. New students and new subjects are constantly entering the picture. We cannot stamp out mathematical illiteracy once and for all at any time in the future.

Second, setting threshold goals in education implies that it is possible to determine a cutoff point, above which one is literate in a subject and below which one is illiterate. Skill in mathematics does not, in any reasonable sense, fit such a dichotomous situation.

The goal of research and development programs in basic mathematical skills and learning should be to improve the skills of all students and adults. I am confident that baseline data such as that called for above will suggest to most anyone that there is room for improvement.

Most of the best known curriculum development and research efforts in mathematics education during the past 20 years have followed the model of substantial grants to projects located at strong institutions -- usually universities. These projects have produced some very good materials and ideas, but the impact on school practice is extremely disappointing.

Any new NIE efforts should explore new R & D models that involve a broader cross section of the mathematics education community, particularly the local school level, in the action. For instance, it seems plausible that many of the research projects I have sketched here could be broadly formulated by a well organized planning group and then developed by more extensive consortia of educational researchers and school systems.

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RESPONSE TO QUESTIONS FOR DISCUSSION AT THE CONFERENCE ON BASIC MATHEMATICAL SKILLS AND LEARNING

E. Glenadine Gibb

This conference is convened to seek some answers to several difficult questions that are being widely discussed by mathematics educators today.

1. What are basic mathematical skills and knowledge?

2. What are the major problems related to children's acquisition of basic mathematical skills and knowledge?

3. What role should NIE play in addressing these problems?

In my attempt to answer some of these questions, I draw on my experience in mathematics education, with particular focus on mathematical learning of children across a wide range of backgrounds and the mathematics education (mathematics and teaching) of both prospective and experienced teachers in elementary schools.

Basic Mathematical Skills and Knowledge

Although computational efficiency is the usual objective associated with basic skills, I believe that there are four areas of basic skills in mathematics: (1) understandings of mathematical concepts and techniques of computation, (2) skill in using these understandings in computation, (3) skill in problem solving, and (4) skill in thinking creatively. Furthermore, these basic mathematical skills should be learned in a balanced program, where one skill is not emphasized to the exclusion of the others.

I have not attempted to make a list of what everyone should know in mathematics. Rather, I take the position that all students have the right to learn mathematics according to their individual capabilities and must be given support so that they can learn; that roads should be kept open in making decisions for their careers; and that their education in mathematics should provide as many career options as possible.

Major Problems in Children's Acquisition of Basic Skills and Knowledge

Without the evidence to judge some problems more important than others, I have identified some practices and conditions that seem to thwart children's acquisition of basic mathematical skills.

1. Children are given insufficient opportunity to abstract and internalize a concept according to their own individual styles of learning before they are expected to put those thoughts in writing using mathematical symbols. Evidence from interviews with children supports this contention. Careful guidance seems to be lacking in helping students symbolize in writing the thoughts they can express orally. Often students are expected to manipulate symbols that have little meaning for them. Thus, resorting to memorization of rules to get correct answers seems the only alternative to these children.

2. Children lack guidance in helping them connect new ideas to those already acquired. Evidence confirms that many teachers -- at least teachers in elementary schools -- closely follow a textbook. If we examine present day textbooks, we note that some efforts are made to provide "connectors." There are constraints, however, -- especially in identifying meaningful "connectors" appropriate for each student using the book. This guidance can best be assumed by the teacher, skilled in the art of questioning and with firsthand information on each student's knowledge and skills. We cannot expect children or even adults to make these connections consciously on their own. As a result, bits of knowledge get stored in a haphazard fashion and their retrieval becomes difficult, if not impossible.

3. Children get feelings of frustration and inadequacy from insufficient experience in developing understanding, followed by ineffective and insufficient practice or drill, followed in turn by little or no opportunity to use their learning (inadequate as it may be) to solve problems (realistic or unrealistic). These feelings can be expected, especially when students are adapted to mathematics programs rather than the programs adapted to the students. In the words of one student: "Just about the time we think we understand, we move on to something else. We quit! Where are we going to use it, anyway?"

4. Curriculum developers and teachers sometimes have unrealistic expectations about student's capabilities. Adults, in roles as teachers or curriculum developers, are inclined to be influenced by students' conversations or to generalize conversations of one student as appropriate for all students. At all levels, we are failing to see through the eyes of our students. Consider some characteristics of the developing child. Young children, in particular, are good mimics and have a strong desire to please adults. For the most part, they are good memorizers. We must carefully listen to children's speech and carefully observe their actions to determine whether what they say is based on understanding and knowledge or is indeed memorized. They may well be trying to provide the answer they know the adult wants to hear. Adults can bias a child's response without realizing it, or can interpret a response in light of preconceived expectations.

5. Alternative teaching methods have not been sufficiently encouraged and explored. One method of teaching is not necessarily the best method for all students or in fact the best for all teachers. From classroom observation, the "show-and-tell," "present," or "explain" methods seem to be the prevailing strategies used. This greatly reduces -- in fact, fails to encourage -- creative thinking on the part of the learners. Needless to say, it is difficult for a teacher to pace so-called information giving for students, whether individually or in small or large groups.

We recognize that an approach used by one teacher may produce quite different results when used by another teacher. I believe that teaching has artistic aspects that are not reducible to a science.

6. We continue to be too concerned with the product, and not enough with the process of learning. Commonly, the "right answers" have been the measures of achievement. This has been a long-entrenched attitude, so it seems, in the education community. The increased use of behavioral objectives to identify expected performance tends to place further emphasis on products. Goals are helpful in directing our attention to outcomes of learning in contrast to "covering so many pages"; but -- depending on their open-endedness -- stated objectives can be most restrictive. The emphasis on the product of learning (correct responses) is reflected in many programs for individualized instruction in mathematics. When children sense that what's important is finishing one unit in order to start another -- or the rate of learning -- they are not interested in sharing information or in describing thought processes. The goal is to "fill in the blanks." Teachers in management roles are often unable to take the time to listen to students or to observe their work. Misconceptions that have prevailed in a child's thinking may be uncovered only after repeated failure.

7. Something is shattering students' self-confidence about their success in mathematics. The feeling that mathematics is something they are doomed to fail can soon cast a pall over students' aspirations to succeed in acquiring mathematical skills. The predetermined ideas of teachers, parents, other adults, or even peers can quickly identify the skills a student has not mastered. It is amazing to observe the change in students' attitudes when teachers emphasize what their students can do rather than what they cannot do.

Identifying Problems

As we continue to try to identify the problems in mathematics learning, we should ask some hard questions:

1. What are the appropriate proportions for drill and understanding? The controversy of drill and practice versus understanding (including the use of hands-on and laboratory types of learning experiences) is a long standing but unnecessary one. I believe that neither can be considered in isolation from the other. A lot of rote drill and practice in the absence of understanding or useful application does little to promote computational efficiency. Likewise, efforts for developing understandings alone are not effective unless they are tempered with drill and practice to build proficiency in computation, in problem solving, and in thinking logically.

2. What levels of mathematical competencies should be expected of today's citizen? How do we assess such competencies? Concern has been expressed for guidelines to sort mathematical skills into the essentials, the desirables, and the optionals, or luxuries. We are concerned about what levels of attainment students should be expected to reach. The assessment of competencies has new interest for teachers of mathematics, especially in light of societal demands for accountability in education.

3. What effect can mathematics have on reasoning as well as on stimulating creative ability in order to enhance peoples' lives now and in the future? Problem-solving capabilities, including some involving quantitative thinking, are required of responsible citizens in today's world. We recognize the limitations of mathematics as a problem-solving tool. But teachers seek assurance that selected experiences in the school setting will at least give students the skills they need to function in today's world -- and at best, enhance their future. Problems should be relevant to students' lives and solvable by drawing on a variety of knowledges and skills -- including mathematics.

4. How can we relate mathematical skills to problem solving in general, to consumer knowledge, and to other practical applications? This problem is closely associated with the preceding one. Because students do not necessarily relate to problems based on adult life, we need to give them problems that are realistic while insuring that they continue to develop both skill in problem solving and the needed understanding and proficiency in computation.

5. What effects can we expect the hand-held calculator to have on the mathematics curriculum? Teachers are confronted with deciding how to make the most effective use of the minicalculator in the classroom. More specifically, they seek ways to use it creatively to enhance understanding of mathematical concepts and improve computational techniques and problem-solving capabilities. Will creative use -- after children have developed the imagery for mathematical concepts and processes -- prove the calculator to be an asset among instructional devices in the classroom?

What Role Should NIE Play in Addressing These Problems?

The problem of acquisition of basic skills and knowledge in mathematics seems to me to be only one of the major complex problems confronting education in our schools. This problem might well be resolved by providing teachers with the opportunity to teach instead of burdening them with many service jobs, paperwork activities, and so on. I feel, nevertheless, that if teachers are allowed to teach there are research studies that could be beneficial to them if the results of such studies are translated into implications for classroom learning.

1. Continued research is needed on how children learn basic mathematics concepts and skills. There is ongoing research in mathematics learning. Several progress reports are to be presented at this conference. NIE might provide support for these types of endeavors, with more emphasis placed on basic mathematical concepts and skills for every citizen. Already consumer awareness is a major concern at the national level. The role of the minicalculator as an effective instructional device in the classroom also deserves national direction, with effective dissemination of knowledge, including possible alternatives for curriculum and curriculum reorganization.

Much research has been done at the cognitive level for purposes of improving the logical aspects of the curriculum. These problems seem easier to resolve than problems concerning affective aspects of mathematical learning. As I see it, research is needed on the latter, research addressing the problems of student attitudes and communications between students and teachers to enhance optimal learning.

2. Research is needed to provide guidance for developing effective education programs for teachers, both at preservice and inservice levels, in accordance with teacher concerns at these levels. Teachers must learn how to guide students so that they acquire basic mathematical concepts and skills. Knowledge of mathematics, of their students, and of the art of teaching certainly are necessary in the education of teachers. Yet, what experiences enable the teachers to integrate these knowledges into effective action for guiding the learning of students? Teachers are learners too. NIE might support research that seeks answers in this problem area. As can be observed in my identification of students' major problems in the acquisition of basic mathematical skills and learning, many causes could be resolved by the teachers.

In identifying the two preceding areas, NIE should be encouraged to support coordinated research efforts in mathematics education, encompassing teams of researchers, to investigate relevant factors of a problem through sequences of studies, replications of studies, and alternative solutions. Of course, the identification of significant problems is the first priority. Any such efforts should be based on knowledge already available. Already, there has been too much "reinventing of the wheel" with little or no attention to past efforts. Such practices may be comfortable, but they do not enable us to push toward new frontiers in mathematics education, where one of the goals is basic mathematical skills for every student.

New Approach to Research

A broader view of research is needed, to include exploratory investigations, materials development and evaluation, interpretations of research, a critical analysis of the research that has been done, and the status of present classroom practices. Within the restrictions of financial support, sufficient time is needed for planning, conducting the research, and fieldtesting the implications for basic mathematical skills learning in the classroom, before there is widespread dissemination of the findings. Dissemination should include suggested alternatives and predict possible outcomes for each alternative.

Continued improvement of mathematics programs and teaching is a lifelong job, as is the acquisition of continuing knowledge and know-how in any profession or business. Furthermore, it involves the efforts of many (schools, universities, research and development groups, and government agencies) in pursuing alternative solutions to enable students to acquire mathematical concepts and skills in fulfillment of their right to learn.

Teachers must continue to find ways of adapting mathematical knowledge so that each and every student in our schools may learn basic mathematical skills. We can support teachers in their efforts through meaningful quality research focused on relevant problems and carefully interpreted into appropriate practices for effective teaching and learning in the classroom.

BASIC MATHEMATICAL SKILLS AND LEARNING

Herbert Ginsburg

Introduction

This paper responds to two questions to be considered at a Conference on Basic Mathematical Skills and Learning sponsored by the National Institute of Education:

1. What are basic mathematical skills and learning?

2. What are the major problems related to children's acquisition of basic mathematical skills and learning, and what role should the National Institute of Education (NIE) play in addressing these problems?

NIE desires a variety of approaches to the questions. My own response stems from my experience as a psychologist investigating mathematical thinking in children 4 to 12 years old.

The key phrase in these questions is "basic mathematical skills and learning." According to the NIE memo of August 7, 1975, "'skills' is to be interpreted in the widest possible sense, as a kind of shorthand for abilities, understandings, knowledge, and so on." The intent of this clarification seems to be to raise questions concerning children's mathematical thinking generally. Such an interpretation makes much sense to a cognitive psychologist. But, accepting this approach, we think it necessary to go one step further. We add to the list, lack of abilities, misunderstanding, and lack of knowledge.

The main question we wish to consider is this: What is the nature of the child's knowledge of mathematics? More specifically, what are the intellectual activities that generate, underlie, or are responsible for, the child's mathematical work? These intellectual processes may result in correct or incorrect behavior. They include such activities as accurately remembering that 6 and 4 are 10; computing that 7 and 7 are 14, because one remembers that 6 and 6 are 12 and then counts on 2 more to get 14; counting on the fingers to get the incorrect result that 6 and 4 are 11; or subtracting by consistently "taking away" the smaller digit from the larger (thus, given 43 - 28 = ? one gets 25).

It should be clear that in our usage, intellectual processes do not refer to behavioral outcomes, accomplishments, products, or objectives. Thus, saying the counting numbers, or solving addition problems are both correct behaviors -- outcomes, products, outputs, responses, accomplishments, etc. -- but they are not themselves intellectual processes. The latter are mechanisms that permit or generate correct (or incorrect) behavior. Thus, intellectual processes are not behaviors in the ordinary sense of the word.

Given this cognitive approach to the question of basic mathematical skills, the next problem is to produce a detailed account of what they are. Our own research (Ginsburg, 1975a, 1975b), conducted over the past 5 years, and that of some other cognitive psychologists, has attempted to do this for the case of arithmetic. Perhaps the results, although limited to this area, are of some general value with respect to questions of mathematical thinking and education.

Mathematical Skills and Learning

In response to NIE's first question (on the nature of skills), consider the following findings and propositions:

1. Before entering school, and outside the context of formal education, children develop various techniques for solving quantitative problems. For example, by the age of 4 years, children can easily see, without counting, which of two sets, randomly arrayed, has "more" (see Figure 1) than the other, at least when relatively small numbers of elements are involved. Children solve the task by comparing the relative areas occupied by each set. This technique is reasonably effective because area is correlated (imperfectly) with numerosity.

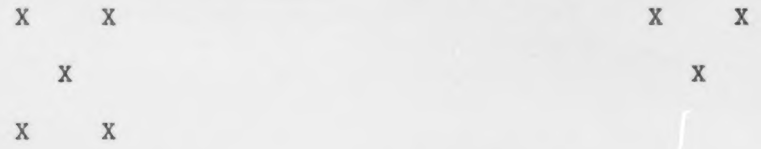

Figure 1. Two randomly arrayed sets.

While such informal mathematics can be useful, it obviously suffers from severe limitations. The absence of precise quantitative tools -- the counting numbers -- leads to a limited field of success. It is possible to see that 9 is more than 6, but not that 42 is less than 43.

2. Perhaps in response to this situation, and without the benefit of schooling, the child develops a practical arithmetic -- counting-based techniques for dealing with quantitative problems. For example, if two sets of objects are to be added, the child learns to represent each element by a finger and to determine the sum by counting. Historical and anthropological research shows that counting-based techniques, usually involving fingers and other parts of the body, can be extraordinarily successful for a variety of arithmetic purposes. Formal education is not a prerequisite for practical arithmetic. Many societies have highly effective arithmetic procedures and no formal education or written mathematics.

3. Informal mathematics of both types -- that is, both without and with counting -- represent generally unsuspected cognitive strengths in children who otherwise have difficulties with mathematics. This is perhaps one of the most central findings of cognitive developmental psychology: On entering school, children possess an imposing array of spontaneously developed and powerful intellectual processes on which education can build.

4. When the child is exposed to symbolic, codified mathematics in school, he or she sometimes learns the standard algorithms as taught. These standard algorithms are important; but they represent only part of the child's skills. Our research shows that the child develops invented procedures, usually based in some way on counting, to perform calculations and other academic tasks. Invented procedures are not necessarily original mathematical inventions or discoveries. Rather, they are techniques the child develops, usually from previous knowledge. An extremely simple example is as follows: A child correctly solves the problem 12 + 6 = ? His answer is "18. Because 10 plus 6 are 16, add 2 more to it and it is 18." This was not the way the child was taught to do it in school. Rather, the child rearranged the problem so as to draw on previous knowledge (the easily remembered fact that 10 + 6 = 16) and to make use of simple calculations (16 + 2 = 18). We have many examples of invented methods, many of them elaborate and effective.

5. Children's errors usually result from organized rules. Errors are not chaotic, nor is it useful to think of them as deriving from such vague mental entities as inadequate "intelligence" or low "mathematical aptitude." Rather, errors have a systematic basis in intellectual processes. Some mistakes are the result of correct procedures incorrectly applied, as when a child uses the standard algorithm, without modification, to add a column of numbers incorrectly alined for that algorithm. Other mistakes are the result of an incorrect procedure systematically applied, as when the child always subtracts the smaller digit from the larger (e.g., 41 - 23 = 22). Children's mistakes seem to derive from strategies of one kind or another.

6. There are discontinuities among different areas of children's work. The most common type of discontinuity involves a gap between the child's written arithmetic and his or her informal, counting-based procedures or invented strategies. Usually the latter are more sophisticated, powerful, and accurate than the written arithmetic that the child is supposed to learn in school. Many children can add on their fingers but not on paper. Counting or mental calculation is often more effective than written work. Indeed, within the area of arithmetic, counting seems to be the most basic skill.

7. Thus far we have discussed various types of strategies (informal arithmetic with and without counting, invented procedures, strategies leading to error, and discontinuity among strategies). We have said almost nothing about how the strategies are learned. The first point to make is that there are many kinds of learning and very little is known about them, particularly those occurring in settings like schools. Consider several types of learning.

a. The learning of informal mathematics seems to proceed spontaneously in the child's natural environment. It seems clear that children all over the world spontaneously acquire various aspects of informal mathematics, like the basic Piagetian concepts, at least through concrete operations. Exactly how this occurs is something of a mystery. But mathematics educators need be less concerned with how spontaneous learning occurs than with the remarkable fact that it does occur. It seems important to recognize that the school child is quite capable of learning on his own; indeed, one can argue that some of the most important things the child learns -- like language -- are acquired almost entirely outside school.

b. One basic kind of learning involves the perception of mathematical structure in school. The child must learn that mathematics is concerned with important regularities and that numbers behave in orderly ways. The world of numbers is highly structured and the child must learn to perceive that structure. Doing so facilitates problem solving and calculation. The child must learn to see that 3 + 4 is equivalent to 4 + 3, in part so that he need not calculate the second sum if he knows that first. Research shows that children only gradually learn to perceive basic regularities of this type.

c. Another basic kind of learning involves the development of a harmonious integration of various areas of knowledge. We pointed out in proposition 6 that there are often discontinuities between the child's approach to written work and his informal or invented strategies. To achieve a fuller understanding of mathematics, the student must learn to eliminate such discontinuities. He must discover that what he already knows is relevant to the symbolic mathematics to be learned. In a sense, the child must learn to trust himself and so must the teacher. Thus, understanding may be considered a comfortable integration between what the child must learn and what he already knows.

d. Another basic kind of mathematics learning is the overcoming of fear and other negative emotions. Many children and other people have an extreme dread of mathematics (Lazarus calls this "mathophobia"). Whatever the reason for the fear, its existence in most people makes the learning of mathematics much more than an intellectual process. This much is obvious and we require no research to prove the point. What is not obvious is how the fear can be overcome, especially in those who have endured it for a long time. We have presented a case study (Stacy, in Ginsburg, 1975c) that attempts to show that, for at least some children, the dread can be overcome by encouraging the use of informal and invented strategies -- e.g., finger counting -- with which the child may be comfortable.

e. Contrary to interpretations of Piaget's theory, young children's learning is not necessarily concrete, nor does it necessarily involve the manipulation of real objects. For example, our research and that of others shows that children at around the age of 4 enjoy the self-imposed exercise of attempting to discover the rules underlying the spoken counting numbers, just as they attempt to infer linguistic rules from speech. Neither of these activities is nonabstract nor does either involve the manipulation of real objects (unless words are considered such).

Problems in Acquiring Skills and What Can NIE Do?

The answer to the first question presented some characteristics of "basic skills." Next we answer NIE's second question: What are the major problems related to children's acquisition of these skills; and how can NIE address these problems? We interpret these questions as asking for an analysis of the main unanswered questions concerning children's mathematical thinking and for suggestions concerning research and other activities for which NIE might provide financial support.

1. Basic research on informal knowledge. We have suggested that before entrance to school, and outside the academic context, children and other people develop informal skills useful for coping with quantitative problems. Yet, much more research in this area is required. For example, it would be useful to know exactly how and to what extent counting-based computational skills develop in "normal" children, in poor children, and in children who later display learning difficulties in mathematics; how such skills develop in the absence of schooling; and how accurate and powerful these skills can be. As we suggested in proposition 7, part d, exploitation of the child's informal knowledge is one strategy for alleviating learning difficulties.

2. Basic research on invented strategies. We have suggested that children often solve school problems by means of invented strategies -- combinations of procedures that children develop from previous informal knowledge, techniques learned in school and, indeed, any other procedures that work. We required detailed description and analysis of such strategies. One wonders how prevalent they are. Do children who usually do badly at school mathematics nevertheless possess useful invented strategies that typically go unrecognized? How powerful and accurate are such invented strategies? They too may serve as a useful foundation for instruction.

3. Basic research on the intellectual processes underlying learning difficulties. There are obviously many children who do badly at school mathematics. What do such difficulties involve? We have suggested two notions that may be useful in interpreting mathematical difficulties and in determining what intellectual processes underlie them. One is that errors have a systematic basis in intellectual strategies. The other is that children's work is often characterized by discontinuities -- gaps between faulty written procedures and more accurate informal or invented strategies.

We have undertaken a series of "clinical-cognitive case studies" (Ginsburg, 1975c) that investigate these notions. The studies attempt to identify, in individual children, the specific intellectual processes that lead to learning difficulties. The studies involve flexible interview procedures and the investigation and treatment of individual children.

We believe that this kind of clinical work deserves to be taken more seriously as a research procedure and to be employed more extensively than it is now. Such research provides a rich picture of individual children experiencing difficulties. The research illuminates their intellectual strengths as well as the bases for their errors.

4. Basic research on learning. Less is known about children's learning than many other issues discussed in this paper. In addition to making the obvious point that more needs to be known, we suggest that research must reflect the diversity of mathematical learning -- informal, perceptual, emotional, etc. A corresponding variety of research techniques is desirable -- e.g., laboratory research, naturalistic observation in the classroom, clinical study. The important task now is the creative exploration of a vitally important activity.

5. The development of new assessment techniques. Anyone familiar with standardized achievement and diagnostic tests should know how bad they are, almost without exception, for the purpose of providing a description of

intellectual processes. They do not yield accurate measures of the kinds of processes that research has shown to be fundamental for mathematical thinking. Hence, we require new assessment techniques. Work on this problem might proceed along several lines.

a. Improved standard testing. Now that research gives some insight into what mental processes should be measured, it may be feasible to develop some useful standard tests. Such tests might be helpful, for example, if they could be used to detect common error-producing strategies.

b. Diagnosis based on clinical interview. Piaget's clinical interview technique is becoming increasingly popular in mathematics education. To maximize the technique's effectiveness for purposes of individual diagnosis, several things must first be accomplished. We require information on the interview technique itself, which has been subjected to surprisingly little study. For example, we need to determine its reliability and the extent to which it is distorted by examiner bias. We require analyses of the theoretical assumptions underlying the technique and detailed observations of its modus operandi. Furthermore, we require a collection of problems that, when presented by means of the clinical interview technique, can provide insight into important aspects of children's mathematical thought.

6. Teacher training. Teaching can obviously profit from insight into children's mathematical knowledge. One useful way in which such insight may be fostered is through teacher workshops focusing on the assessment of children's mathematical thinking and on teaching strategies that derive from such an assessment. In such workshops, teachers might learn to do clinical interviewing; analyze selected TV tapes of children's mathematical behavior, conceptualize the processes of mathematical thinking; consider teaching techniques (e.g., reliance on informal mathematics) that stem from knowledge of children's work.

7. Interdisciplinary centers. NIE can facilitate work of the type described above by establishing interdisciplinary centers for work in mathematics education. Mathematicians, educators, and psychologists -- all willing to work directly with children and each other -- may produce innovative solutions to some of the problems described above. One part of such a center might involve a learning clinic devoted to the study and treatment of children experiencing difficulties with school mathematics.

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PROBLEMS RELATED TO CHILDREN'S ACQUISITION OF BASIC SKILLS AND LEARNING OF MATHEMATICS AND SOME SUGGESTED R & D OPTIONS FOR NIE SUPPORT

David Helms and Anna Graeber

We appreciate this opportunity afforded by the Euclid Conference to offer a few observations on some of the problems that seem to relate to children's acquisition of basic skills and learning of mathematics, and to suggest some R & D options that NIE may wish to support. To the extent that our observations correspond to similar findings of others, they may suggest dimensions along which the quest for ways and means of enhancing mathematics learning might be pursued to some benefit. To the extent that the circumstances that gave rise to these observations are open to alternative interpretations, they should perhaps be foci for disciplined study.

We do not assert that the problems upon which our observations are fixed constitute an all-inclusive set. Neither would we argue that they are the most likely causes of barriers to learning. We do contend that they will require attention in any instructional treatment intended to enhance students' acquisition of basic skills and learning of mathematics. The problems we perceive to be related to children's learning of mathematics are discussed under the "Instructional Program" and "Management of the Instructional Program."

Instructional Program

Basic Mathematical Skills and Learnings

In our work at Research for Better Schools (RBS), as developers, implementors, and evaluators of instructional programs in mathematics, we have been more concerned with problems of instructing than with problems associated with the choice of appropriate mathematics content. Nevertheless, it has been our intention to provide "good mathematics" that is eminently instructable and learnable by learners in grades K-8. Frankly, we have attempted to insure the quality of the mathematics content mostly by selecting it from areas of consensus among major national programs.

The careful analyses employed for the identification of consensus, along with our subsequent experiences developing and implementing programs based upon this content, prompt us to wonder about several aspects of the "modern math" as they might relate to problems of teaching and learning. Mathematics appropriate for inclusion in the school curriculum it seems to us, is a function of (1) content that can be offered to learners at their respective stage of development, (2) relevant learner need for and interest in mathematics, and (3) the time likely to be available for mathematics instruction.

The argument that "...any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (Bruner, 1962) is well known. The condition upon which that statement is predicated is less popularly observed. "The task of teaching a subject to a child at any particular age is one of representing the structure of that subject in terms of the child's way of viewing things." (The emphasis is ours.)

We suggest that one explanation for the failure of new math to help the majority of the national student body to improve performance through better understanding (Gray, 1974) may be the use of mathematical heuristics that do not reflect "the child's way of viewing things." It is our observation that the mathematical heuristics used to elucidate structure and operations frequently have had a contrary effect upon many students. Set notions; associative, commutative, and distributive symbolic manipulations; and computations in unfamiliar nondecimal number bases have perhaps been extended and emphasized beyond their intended use. Still, we wonder about the value of heuristics that require more sophisticated behavior of the student than the learning they were intended to elicit.

At any rate, we propose that the effectiveness of the mathematical heuristics of the new math be studied in terms of the degree to which they promote understanding. We see this as a systematic effort worthy of NIE support. Arguments made for the retention of certain mathematical heuristics (those not justified in terms of their superior enhancement of understanding) on grounds that the mathematics is worthy in its own right are arguments of need, as are those which claim that mathematics provides valuable precursor experience that will benefit learners in some way at a later date. Such claims should be evaluated in the context of competing need claims that suggest other kinds of studies.

We are struck by the reasonableness of Gray's assertion that the two principal concerns guiding overall determination of content needed at every stage of student development should be (1) mathematics necessary for a minimal level of quantitative competence that contemporary culture and its immediate future prospects demand, and (2) mathematics necessary for some intellectual grasp of the mathematical sciences and their applications to our complex world. We would add a third major concern, mathematics relevant to the experience of the learner. All learners are not so "turned on" by the beauty of mathematics as we would hope. Mathematics that is patently relevant to their interests and needs in daily living is more likely to be motivating. To the extent that more esoteric mathematics can be made relevant to learner interests and needs, motivation for study of such mathematics may be aroused.

Apparently motivated by the startling announcement made in 1965 that anything can be taught to learners at any stage of development, many textbook publishers and teachers have responded as if whatever can be learned ought to be taught as early as possible. Economists know full well that the needs and wants of humans are endless, but the means of their satisfaction are limited and they have thus devoted themselves to the problem of maximizing utility through optimal allocation of available resources. If there were no other limitations, the pressures exerted upon learners by the amount of learning expected in the time available for instruction in mathematics should cause us to weigh carefully what mathematics learning is of most worth and when.

Insufficient attention to time constraints makes some students feel bombarded by successive waves of radically new ideas and too little time to gain even a modest grasp of some of them. We are told that feelings of "pressure" frequently motivate some students to declare for themselves unofficial holidays from school.

Given the fact that some people are prone to prescribe what learning is of most worth solely on the merits of the mathematics and the capability of the students at specified stages of their development, it should be noted that sufficient time allowances need to be made for both developmental activities and practice if the learners are to acquire the predicted learning. A number of studies point to improved student achievement when between 50 and 75 percent of the class time is devoted to development activities. (Reidesel and Burns, 1973). Yet, if too little time was spent on development activities in previous years, it is our observation that too little time has also been spent upon practice in the recent past. The result of the latter has been that learners spend discouraging hours on problems they could quickly solve if they had a ready grasp of the arithmetic facts.

At the risk of overloading an already difficult content determination, our experiences nevertheless tell us that not only does the kind of mathematics that can be offered differ according to the developmental stages of the learners, but learners also differ in terms of their positions within the developmental stages and in terms of the relative amounts of developmental activities and practice they require. This suggests that the study of what content should be offered, as well as when, and for how long, should, in addition, consider how the determination might be adaptive for variations in competency and need among individuals.

It is proposed here that the determination of what mathematics is most worth learning is a task that will require careful and systematic study from the perspectives of several interest groups. In this respect, Willoughby's comments in the Proceedings of the Conference on the Future of Mathematics Education (1975) deserve attention.

> The process of setting goals...probably should be overseen by a distinguished commission of citizens, some of whom are mathematics educators (including classroom teachers), mathematicians, natural and social scientists, humanists, consumer advocates, and other representatives of the society at large.

It is grossly unfair to impose on already overburdened teachers the decisions implied by the questions we have raised, as seems to be the inclination of today's educational leadership. Gagné (1970) has noted other important and time-consuming tasks that are dependent on teachers for their adequate treatment.

> Predesign of instructional conditions greatly reduces the necessity for the teacher to use valuable time in extemporaneous design, and thus makes possible for a proper emphasis to be restored to the teacher functions of managing instruction, motivating, generalizing, and assessing.
>
> The...point should not be taken lightly, since these other functions deserve a great deal of emphasis in education,

> and are likely to suffer neglect by teachers who are overburdened with the very difficult task of extemporaneous design of instructional conditions.

That the studies proposed here are the proper concern of NIE is confirmed by the prospect that such studies will need to be continuing efforts. A reading of the history of mathematics education in the United States indicates the continued growth of the discipline of mathematics and the technological advances of society are reflected in school mathematics content. The most constant aspect of change in school mathematics -- the inclusion of more content -- presses upon us today. We are being told that we need to teach metrics and problem solving, we should do more with applications of mathematics, we must attend to consumer mathematics, and we need to teach students how to estimate and use calculators. The temptation to once again "add on" to the content is strong. The questions of what mathematics learning is of most worth, when, under what conditions of time, and for whom obviously require continuing resolution.

The studious determination of what mathematics is of most worth under the priorities and circumstances of the time will likely be considered the basic mathematics skills and learnings of that period.

Strategy and Materials of the Instructional Program and Problems That May Raise Barriers to Learning

Whatever the mathematics program, we are convinced that instruction can only be effective to the extent that the teacher is successful in achieving appropriate matches between learners and their instruction. Glaser (1967) notes the following minimal instructional tasks that must be systematically attended to if "good fit" matches are to be consistently arranged for learners: (1) specification of expected learning, (2) preinstructional assessment of learner competency and need, (3) provision of appropriate learning experiences, and (4) postinstructional evaluation of learning.

The difficulty for teachers in arranging appropriate matches lies in the differences that distinguish learners. Logically, critical differences among learners imply that the instructional tasks should be attended to for the learners on an individual basis. Failure to do so, we observe, tends to cause problems that inhibit learning of mathematics.

Persistent mismatches between students and their mathematics instruction lead to cumulative boredom or defeat and eventually to disruptive classroom behavior and/or absence from school. In every case, the learner's self-concept as a mathematics student is likely to be less appropriate than it could be, and, in every case, learning by the student likely will be less effective than it might be.

On the other hand, it is our experience that frequent achievement by students of demonstrable success in mathematics, positively reinforced, is a most powerful motivator of mathematics learning. Continuous progress seems to occur, according to our observations, when (1) students possess the necessary prerequisite competencies; (2) the transitional instruction provided students is appropriate for the mathematics to be learned and for the individual needs of the students; (3) specific feedback and reinforcement are accessible to the students;

(4) the criteria for success are objective and clearly understood; and (5) the time for testing the students' acquisition of learning is determined by them.

For us, the keys to effecting good matches between learners and their mathematics instruction are (1) appropriately specified and adequately sequenced curriculums of generous scope; (2) criterion-referenced assessment instruments correlated with the performance objectives; (3) self-instructional materials; and (4) specific preparation of teachers for the management of adaptive instruction. The first three keys warrant special comment to reflect design progress we have made since the early days of IPI Mathematics. The fourth item will be discussed in the next section.

Our recent work has confirmed our suspicion and fond hope that specifying and sequencing mathematics performance objectives does not require trivial objectives, forfeiture of exploratory and inductive experiences for learners, deprivation of student content choices, isolated study, or exclusion of enactive learning. It taught us that accommodation of these attributes does require greater imagination, more creativity, and sharper designer skills than we had supposed necessary.

That the kind of instructional program we describe may be positively related to significant improvements in student learning of mathematics is suggested by the findings of Holzman and Boes (1973), in a study they conducted for the United States Office of Education. According to these researchers, eight common characteristics of successful compensatory education programs included: (1) clear objectives stated in measurable terms and supported by instructional techniques and materials closely related to the objectives; (2) attention to individual needs, including careful diagnosis and individualized instructional plans; (3) a structured program approach that stresses sequential order and activities and frequent, immediate feedback. The criteria for identifying a successful program were student achievement and attendance, positive self-concept, and fulfillment of physical needs.

Management of the Instructional Program

Barriers to learning mathematics are not likely to be reduced by programs alone -- at least not by programs with which we are familiar. Programs provide means, procedures, and accommodating materials to facilitate the teacher's attention to the tasks of instruction for or with the students. How well the tasks are attended to likely determines the probability of success for students and their attitude toward learning. The ideal combination would seem to be well-motivated students with a well-designed, well-managed mathematics program.

In our work with teachers, helping them to shift from standardized attention to the tasks of instruction for the class as a whole to adaptive attention for individuals, it is our observation that many opportunities exist for preservice and inservice improvement of the teacher's program management skills. Needless boredom and frustration of students could be avoided if more teachers had a better grasp of mathematics content and necessary diagnostic skills.

For many teachers, knowledge of mathematics is too meager and diagnostic skills too deficient to recognize the patterning of errors that reveal the nature of student misunderstanding. Too often, lack of versatility in mathematics and one-to-one instruction causes teachers to miss opportunities to provide just the right enactive or iconic experience that might pierce the barrier to a student's comprehension. Many options revealed in teacher-student exchanges urge the use of adroit questioning to move the student to higher order reasoning, but these are missed for want of both mathematics and tutorial skill.

Programs do not provide teachers with these skills, only the opportunities for their use. If learning of mathematics by students is to be enhanced, teachers must be helped to develop their own mathematics competence and, particularly, their skills to diagnose difficulties and prescribe a variety of appropriate alternative learning solutions. Moreover, their preparation to conduct these activities should focus on attention to individual students. Learning and erring are individual behaviors. If at times, and on the surface, individuals seem to share the same learning or the same erring, it is likely that they do so for reasons unique to each of them.

Just as important as mathematics competency are the skills of program management that encourage students to pursue learning whether independently or in groups. If the teacher concentrates upon attention to the tasks of instruction for students individually, obviously students not receiving immediate attention must be engaged in self-instruction. Delegating instruction to students requires that they be free to move about and engage in a variety of experiences alone or in the company of other learners. Successful management of such a program requires a high level of trust on the part of both teacher and students. Generating trust in themselves and among their students is a difficult skill to acquire for many teachers. Again, it is presumed that teachers bring such skill to the program.

Attention to the tasks of instruction for students individually is highly compatible with the notion of small group instruction. There are many reasons for learning in small groups, even though emphasis may be on matching individuals with instruction. Several learners sharing the same difficulty, if not for the same reasons, may be more effectively attended to in a group. Safety, practice in communication, and cooperative inquiry are all reasons for small group settings. When emphasis is on "best fit" instruction of individuals, formation of small groups will be predicated upon (1) the need for some kind of learning that is best acquired in a group setting, and (2) all members having the prerequisite learning necessary for success in the group. Arranging and guiding small group instruction are demanding skills that need special attention in preservice and inservice preparation.

It is our observation that many teachers do not possess many of the skills necessary for effective program management. Nor do many acquire them as a result of the mathematics program itself. Since the skills, well performed, are probably positively related to children's acquisition of basic skills and learning of mathematics, we recommend that a careful study be made to determine the necessary program management skills, adequacy of preservice preparation of teachers in these skills, and effective means for their instruction. As with studies suggested earlier, we propose that NIE can provide appropriate leadership in this area.

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SKILLS AND SSKILLS

Leon Henkin

What Are Basic Mathematical Skills and Learning?

A skill is an ability that is always the second half of a two-stage learning process; the first half of the process involves a form of understanding.

We can see this process clearly by observing the acquisition of familiar skills such as the ability to tie shoelaces. In the first stage a child learns the geometric form which the lace is to assume -- a pair of loops built up on a wraparound crossing, tied together in a certain way -- and comes to understand that this form can be achieved by passing through a specified sequence of intermediate forms.

At the end of this first stage, the child, by concentrated effort, can achieve the desired form by going through the learned sequence of intermediate forms, correcting an observed incorrect form by comparison with a remembered correct form. At this stage the child has an ability to tie shoelaces, but does not yet possess the skill.

In the second stage of the process, the nature of the learning changes, and the true skill results. Somehow by repeated tying of laces, the eyes and the fingers learn to perform their proper role without the need for continuing supervision and direction from the part of the mind that was involved earlier, when forms were perceived and compared with remembered correct forms. When the skill is attained, laces can be tied with extremely low probability of error, and during periods when "attention" and memory may be occupied with completely different objects and activities

In Esty's memo to conference participants he reports the use of the word "skills" by the NIE group as being "in the widest possible sense," a shorthand for "abilities, learning, knowledge, and so on." Of course mathematicians can agree to use any word in any sense, as long as it is clearly specified. But to me it seems crucial, in a conference on problems of early mathematics learning, to be able to distinguish and discuss the narrower sense of the word "skills" that I have attempted to delineate.

Thus, if the word "skills" is to be used in the broad sense, we need a new term for the narrow sense. Since the essence of a skill in the narrower sense is the possibility of exercising it at a subconscious level, while the consciousness is occupied with other (possibly related) problems, I propose the term "subconscious skills" -- shortened to "sskills" (pronounced "sss-kills") in the rest of this paper.

When one seeks to make an inventory of the most basic mathematical sskills one thinks at once of the use of standard algorithms to compute values of the arithmetical functions for whole number arguments. Actually, there is still a more basic numerical sskill that tends to get overlooked in school programs because it is thought of as coming at the preschool level -- namely, the sskill of counting a bunch of objects. Arthur Kessner at Berkeley has been investigating the acquisition of this sskill by young children, using various adapted forms of hand-held calculators, and he believes that many problems encountered by school pupils in dealing with addition and multiplication stem from deficiencies in their counting sskills. Furthermore, when one begins to analyze counting sskills one finds that they are built up from a variety of prior sskills, such as chanting the numerals in standard order, touching objects one at a time while chanting.

Mathematico-Linguistic Sskills

It will be most useful if I desist from enumerating or analyzing numerical sskills and instead, call attention to certain other mathematical sskills, extremely basic and important, that often tend to get overlooked in discussions of elementary mathematics curriculums.

There is a whole class of sskills, that might be called mathematico-linguistic sskills, that deserve a great deal of our attention. Perhaps the most fundamental of these consists in assigning simple names to objects in which we get interested. In nonmathematical contexts people normally get to name only their own children -- or an occasional peak if one is an adventurous mountain climber. But mathematicians confronted with a triangle will at once lable its vertices -- usually with letters from the most familiar alphabet. This greatly facilitates all further work in discovering and communicating properties of the triangle.

Although naming the vertices of a triangle is a very simple sskill, there is a whole hierarchy of naming sskills that reaches very sophisticated levels in attacking combinatorial problems. To give a very elementary example involving triangles again, there are several methods for distributing names to the sides of a triangle in a way related to the names assigned to its vertices, which greatly facilitate discovery and exposition.

The use of ad hoc names, especially letters, is normally found only at a late stage in the mathematics curriculum -- namely, when algebra and geometry are developed. However, because the bulk of linguistic sskills in nonmathematical contexts are required much earlier, it is reasonable to reexamine our curriculum to see whether mathematico-linguistic sskills could not also be profitably advanced. We know that certain kinds of activity may be rejected, or undertaken reluctantly, at a certain age, because they seem excessively boring; yet the same activities may seem interesting and attractive at a younger age. This is especially true of the repetitive tasks that are common in acquiring linguistic sskills.

Among the mathematico-linguistic sskills I include certain simple uses of the connective words, "not," "and," "or," "if ... then." For instance, if we know that today is not Tuesday and not Friday, then we can conclude that it is not true that today is Tuesday or Friday. Again, if we know that today is Wednesday or Sunday and we learn that today is not Wednesday, then we can conclude that today is Sunday. Obviously, I am talking about simple forms of logical inference.

Use of Logic in Mathematical Study

I know that many educators and mathematicians look upon logic as a highly abstract subject entirely unsuitable for elementary mathematics students -- and indeed I am among them, if we are speaking of formulating general laws of logic, and committing them to memory or studying their interconnections. On the other hand, because the words "and," "or," and "not" are accepted as basic parts of the working vocabulary of every school pupil, specific inferences (such as those involving days of the week that were formulated above), can be justified and understood directly in terms of the meanings of these three simple words. My former student, Nitsa Hadar, has shown conclusively that fourth- and fifth-grade students can greatly improve their ability to separate valid from invalid inferences, based on the meaning of sentential connectives, through a short unit of their mathematics course taught by their regular teacher. It is possible that this result could be obtained at an earlier age.

It is my belief that the understanding of simple logical inferences by elementary mathematics students, which we now know how to accomplish, can be made the first stage of a process by which a true sskill at making inferences is attained. If substantiated, this thesis could greatly enlarge the potential scope of elementary mathematics curriculums.

Developing Skills in the Wider Sense

I have been writing about sskills, but it is time now to turn to skills in the wider sense delineated by Esty, that is to say, all sorts of abilities involved in the learning of mathematics. One that comes most quickly to mind is the ability to integrate computational sskills into the process of problem solving.

For an example of problems that require the use of arithmetical sskills, consider problems that require the solution of algebraic equations. For example, in solving quadratic equations it is often desirable to express a quadratic polynomial as a product of linear factors, and students are furnished with a good supply of problems in which all coefficients involved are integers. To factor a quadratic polynomial in such a problem, pupils are taught to seek the numerical factors of the first and last coefficients. How is this achieved?

Normally, the pupil considers the absolute value $\underline{a}$ of one of these coefficients, successively tests each positive integer $\underline{b}$ less than $\underline{a}$, and by appeal to knowledge of multiplication facts or sskill in applying a division algorithm, the pupil determines whether or not $\underline{b}$ is a factor of $\underline{a}$. To deal with both first and last coefficients in this manner requires, on the average, testing some eight number pairs, to determine whether the smaller is a factor of the larger.

It is quite impractical to expect a pupil to factor any sizable number of quadratic polynomials in this way, with a high degree of success, if the testing of each number pair requires concerted effort. In other words, the application of basic multiplication facts or of a simple division algorithm must be reduced to a true sskill, in order to free the "thinking part" of the mind to cope with algebraic parts of the problem that lie beyond arithmetic.

Every teacher knows that the "word problems" are always the most difficult part of any elementary mathematics textbook. The failure of a sizable number of research efforts to devise mechanical systems of translating from one language into another, emphasizes the unlikelihood that translation of word problems into mathematical symbolism can ever be reduced to the level of a sskill -- it will always be an ability that involves the concerted attention of the "understanding part" of the mind. I view it as another dimension of mathematico-linguistic skills (one "s").

Although some mathematics educators have emphasized problem solving as "the key" to improving elementary mathematics, it seems to me to be only a limited part -- albeit a very important one -- of what is needed. In broadest terms, the desideratum is the integration of mathematical learning into the totality of human experience. Although this phrase has a grandiose and unrealistic sound, I believe it can be brought down to earth in ways that are realistic in the context of our elementary schools.

The essence of this concept is the discovery of regularities in limited portions of our experience through the formation and testing of conjectures. Obviously this is scientific method in the broadest sense, and if pursued in the school setting will lead to the development of mathematics in a context of handling and observing physical objects. This at once points to a variety of sskills, and more general skills, involved in recording and systematically arranging the results of observation. Beyond these lie the complex of skills needed to pass from observation to abstraction, from physical manipulation to symbolic manipulation, and from symbolized abstraction to application.

Finally, it is essential to inculcate the habit of guessing, and to improve guessing in specified areas through the development of local intuitions. The activity of guessing is almost "opposite" that of the systematic application of prescribed algorithms. Consequently, it has actually been discouraged by some teachers whose concept of mathematics is bounded by the traditional elementary curriculum. Yet guessing is at the heart of mathematical activity, both pure and applied. We must find ways of making our students good at it.

General Appeal for All Students

Let me emphasize that the ideas expressed above seem relevant for all elementary students, not only for those who will develop special interests in mathematics. The problems of poorly prepared, low achieving students, deficient in motivation, have justly been perceived as a facet of social changes deserving of our most intense efforts. But we must tackle these problems without sacrificing our aspirations for the students involved. Certainly a sound system of elementary education must meet the needs of those students who will not go on to higher education, but to fulfill the democratic impulse in our national life it is absolutely essential that no social class of students be precluded, or even inhibited, by our system, from seeking the highest educational levels.

What Are the Major Problems Related to Children's Acquisition of Basic Mathematical Skills and Learning, and What Role Should NIE Play in Addressing These Problems?

My response to this question will be rather shorter than what I have written in the first part of this paper -- mostly because I have many fewer credible ideas in this area.

A natural starting point is the ability level of elementary school teachers to understand and to teach mathematics. While there are teachers who excel in this area, the great majority of teachers are poorly motivated and poorly prepared, both to understand and to teach mathematics.

What can NIE do about this problem? Because its mission does not include mass teacher improvement, I conceive that it should seek to devise mechanisms for utilizing other national resources to effect such improvement. Taking note of Esty's comment that NIE is interested in both immediate practical solutions and long range questions, such efforts could proceed along two tracks.

First, there should be explored and compared a variety of low cost means for supplying continuous inservice help in mathematics to elementary school teachers by promoting working relations between them and secondary school or college teachers.

Second, means should be sought to encourage mathematics graduate students to involve themselves with questions of elementary mathematics education, including classroom experience, and to encourage school districts to employ such persons in specialist positions designed to stengthen mathematics teaching in all elementary classes.

At the University of California, through such programs as the Community Teaching Fellowship Program, the Peer Teaching Project, and a teacher development program organized by the Madison Project, I have observed and worked with many mathematics graduate students who have been drawn into work in the elementary schools. These efforts have been very helpful to all three groups -- the young mathematicians as well as the pupils and teachers with whom they work in the schools. Some of these graduate students have shown a sincere interest in a career that would allow them to continue working in the schools. But only two or three, so far, seem likely to surmount the bureaucratic obstacles to such an unconventional arrangement and to fulfill their sense of commitment.

Along with the problems of teaching mathematics, the most commonly observed problem in the elementary classroom is that of motivating students to study mathematics. Educators tend to divide into one class saying we must show students how mathematics is relevant to their own lives and interests, and another class which stresses the need to recast classwork and exercises into forms that are fun -- often games -- or intrinsically challenging. Obviously there is nothing contradictory about these two perspectives. Certainly NIE can sponsor research motivation, seeking to find strong motivating factors -- both of the "relevance" and the "fun" kinds -- that lend themselves to various types of mathematical activity, and that are useful to significant classes of students (defined by such factors as age, sex, ethnicity, and socioeconomic class). The problem for researchers to find a useful and reliable measure of the motivational impact of specific forms of mathematical presentation, problem, or activity, will be a challenging one. A third aspect of motivation is clearly linked to the level of interest and enthusiasm that teachers show toward their mathematics work in class, which brings us back to the original problem.

Although the problems of improving teacher competence and the motivation of students are immense, and though progress in their solution is quite necessary to improve significantly the teaching level of mathematical skills and concepts, there are very different kinds of problems whose solution is equally important, and these perhaps provide a more fruitful area for NIE-supported work. What I have in mind is adding to our understanding of how individual students come to develop mathematical concepts and related sskills, through experimental programs that combine the research efforts of psychologists and mathematicians.

For an example of the kind of work that could be undertaken, let us revert to the sskill of tying shoelaces, considered earlier in this paper. This humble, practical task is not normally thought of as a mathematical activity, but it certainly has a mathematical aspect -- as is readily apparent if one broadens the activity to include learning to tie and untie a variety of knots, or determining when certain intertwined pairs of looped laces can be separated. In fact, at Berkeley's Lawrence Hall of Science a successful children's class was built around exploratory play with "topological puzzles" made of string and wood.

It was through personal observation and children's behavior in dealing with these puzzles that I completed my formulation of the two-stage process for acquiring a sskill (described in the first part of the paper). I realize keenly that from the research viewpoint this is only a very preliminary hypothesis -- but it is the kind of hypothesis that can provide the framework for a host of related, empirical studies.

For example, if acquisition of a sskill proceeds through a first stage of understanding with the attentive mind, followed by a passing of the task performance to a subconsciously directive portion of the mind (akin to distinction between voluntary and involuntary muscle control), how can teachers recognize when a student is "ready" for the second stage? We must experiment with the development of readiness measures, compare the total times of sskill acquisition and longevities of sskills when second-stage teaching efforts are begun early or late in the readiness period.

We can investigate whether there is a hierarchy of sskill levels, some very simple sskills being developable in a single-stage process and others requiring even more than two stages. With respect to particular mathematical sskills we may try to devise a system of classifying several distinctive "learning styles" that may be common among school pupils, and if we succeed we can then seek ways for a teacher to identify the style of a given pupil, and we may devise a distinctive teaching strategy correlated with each learning style.

If successful, fundamental investigations of these kinds will provide a bedrock upon which a host of practical techniques will be based, aimed at improving teacher performance, classroom materials and books, and curriculums themselves. Of course some general efforts of the kind I have attempted to sketch have been undertaken, but from my admittedly inadequate acquaintance with them, my impression is that their design has too often failed to allow for the diverse mathematical ways that exist for dealing with even the simplest notions of arithmetic or logic. For this reason, I stress the importance of forming research teams in which mathematicians and psychologists cooperate.

A GLOBAL APPROACH TO MATHEMATICS INSTRUCTION: AN EXPLORATION

Norma G. Hernandez

The ostensible reason for our participation in this conference is to attempt to identify basic mathematics skills to be taught in elementary school. I believe that our task is larger than that, in that we need to determine what a skill is, and identify a philosophical position relative to what content should be taught. This position should have the power to assist in the generation of basic skills. Method must also be considered in this connection, otherwise the assigned task cannot be performed and the product will not carry impact or be fruitful.

The purpose of this paper is first, to explore briefly one view of the meaning of skill, and second, to present a position relative to the content that is to be taught in elementary school mathematics. The scheme for identifying content will provide a structure for identifying specific basic skills. The content will then be identified together with an associated method of instruction. Finally, a number of ideas worthy of research and methods for conducting the research will also be suggested.

It may be of interest to note the factors that motivated the approach taken in this paper. The problem before us is to identify mathematical content for use by elementary school students in general. However, it must be pointed out that large numbers of students in the nation have problems of learning in a language other than their first language. The implications for the learning of mathematics are clear. Because this segment of the population has not achieved at a "normal" level in the past, steps must be taken to consider alternatives for the solution of the problem. By using the motivation to help one group of students who have "different" learning problems than the general population, it is hoped that other persons having associated learning problems will be helped. This approach will permit us, as educators, to broaden our view of student needs, styles, and outcomes.

Definition of Skill

At the present time, the world in which we live is sufficiently complex to require that we define skill as more than mere technical competence. It is suggested that the term be defined not only as knowledge of the means and methods of accomplishing a task, but as one English dialect uses it, to understand or comprehend (Webster's Dictionary, 1971). Hence, it would require skill to develop the concept of algebra as combining, by use of symbols in an equation, mathematical entitites in accordance with assigned rules. Skill, in the latter sense, would include the use of cognitive operations at the concept acquisition level.

The motivation for this position comes from a conviction that "concepts" at varying levels of significance, maturity, and depth, can be learned at all ages.

from birth to maturity, by people of nearly all intelligence levels, provided there is the capacity to develop "speech."[1] If this assumption holds, then all school content should include the teaching of cognitive processes that include higher levels. No child, or group of children, should be classified or categorized for the purpose of receiving instruction strictly at the lower levels of cognitive processes. This includes discrimination on the basis of age, sex, ethnicity, or "intelligence level." This suggests that new methods must be developed for reaching students who have been excluded in the past on the belief that age, sex, ethnicity, and/or IQ solely determine capacity for learning concepts. In addition, we must develop in children the capacity to learn how to learn for all disciplines. In a broad sense, learning to learn that is discipline specific. Learning to learn mathematics, I believe, can be taught as a skill, in the broadest sense.

It is thus assumed that all people of normal intelligence can learn at various levels of cognitive processes. The position taken in this paper, then, is that all instruction on developing skills in elementary school mathematics should address itself to the facilitation of skills that include the higher cognitive processes, namely concept development.

Two Approaches to Method -- Two Approaches to Content

It is important to consider a position relative to what should be taught in mathematics before undertaking the identification of specific skills. By this I mean that we should decide what knowledge is of most worth, in order to specify required skills. If this position is not taken, only a random and perhaps incomplete list of objectives will be identified. A principle or method for the selection of skills should be identified. This method should be used to generate a comprehensive and complete list.

It is proposed that methods of instruction be considered as a possible source of ideas for identification of content. It is not the purpose of this presentation to suggest that method should determine content. What is suggested is that method may inadvertently determine content if the instructor is not aware of the possibility. More importantly, however, is the suggestion that the demonstration of a method of instruction that is significantly different from the one currently used may signify that different content is being taught, or at least emphasized.

In identifying possible avenues for the determination of content, tentative conclusions may be made by reasoning backward. It may be that if two pedagogical styles can be identified, it is possible to associate with each a particular cognitive style. This cognitive style may carry with it its particular method of selection of content, tending to emphasize, or deemphasize, certain notion. If this is the case, as suggested by some researchers (Ramirez and Castañeda, 1974), it is possible to identify an avenue by which different views of content may be generated. These various methods will, in turn, appeal to students who have corresponding cognitive styles. It has also been suggested that culture plays an important role in the development or manifestation of these styles (Lesser, et al., 1965).

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[1] It is merely suggested that the learning of speech may be one measure of a minimal capacity for learning concepts at a basic level.

It is not generally recognized that a particular teaching method may determine content. Two general methods will be discussed -- the analytic and the global. The former will be discussed first because it is the one currently in use; its relation to the generation of content will be explored.

Analytic Method

Upon examination of the leading texts in elementary school mathematics and secondary texts in particular, one finds that the approach taken is that of teaching particular skills that lead to the acquisition of skills of symbol manipulation in the shortest time possible. A large number of ideas that lead to the acqusition of skills mentioned above are presented. This approach is labeled analytic in the sense that the content is presented in a particularized manner; the minute details of the topic are learned first and the main ideas are presented subsequently.

In the analytic approach to addition, for example, the emphasis is on learning facts and using algorithms for the four operations without giving students an opportunity to develop an understanding or even a sense of what an operation is. This approach implies that the higher levels of understanding will come about later, in subsequent grades. This is very rarely the case, and hence, the content is determined.

Another example is in the teaching of algebra that begins in many cases with students learning the rules of punctuation (e.g., marks of inclusion, manipulation of symbols, etc.) in equations, without understanding how it fits in with the general notion of algebra. In this approach it is possible to emphasize manipulative skills to the exclusion of basic concepts; the emphasis in such an approach is on the lower level of cognitive processes, by default. The cumulative effect of this approach is that cognitive processes of higher order are taught only at advanced levels.

Let us examine a case in point. Modern math has been taken to task because, supposedly, concepts were taught and skills forgotten. My opinion is that concepts were not taught through a process of concept development, but as "facts" that can be committed to memory and manipulated. In this approach, if a concept is a fact it is static and cannot grow, change shades of meaning, or be used as a tool for problem solving, as can a true concept. Thus, the intent of modern mathematics, to teach understanding, was not really achieved. The point is that because the traditional approach in our teaching mathematics is analytic, with details emphasized, concepts were taught as facts, i.e., something static. The static concepts, then, could not be used by students to provide the flexibility for problem solving or the understanding of algorithms to facilitate development of skills.

The implication in the case cited above is that the approach to mathematics instruction traditionally has been analytic, with an emphasis in the rote memorization of facts, and that the same approach was used in concept development. It is suggested that other approaches may be more fruitful for teaching concepts.

Global Method

A second approach to the teaching of mathematics is one that will be labeled global. In this approach, the generic overarching ideas of a topic are presented first with increasing differentiation among subtopics and their reconciliation through integrative processes, à la Ausubel. In this approach, concepts are taught first, through advance organizers, say, then details follow where necessary. For example, in teaching addition, one operation would not be taught separate from the others. The "sense" of concept of operation would be taught first, through varied and wide use of examples such as games, manipulative objects, etc., with explicit emphasis on the students' awareness of the methods used to learn. The details, i.e., number facts, would come later. One disadvantage of this approach is that skills could be overlooked, as concepts were in the analytic approach. If care were taken, however, special provisions could be made whereby specific skills viewed as necessary by the student, would be taught, perhaps in a more palatable and humane way.

One way to humanize schools is by providing variety in content and approaches that has potential for appealing to a large number of students. Variety can be provided by presenting mathematics via the two methods proposed above and even by the use of a combination of the two. If our decision is to explore the possibility of the use of two curriculums, half of our task is complete because the traditional method of presentation has employed the analytic or particularized approach. For example, the presentation of the algorithms of the four operations using whole, rational, and real numbers, one at a time, over a period of 8 years is the approach generally used. The suggested approach, the global one, would need to be developed in entirety. There is no example I am familiar with that presents mathematics from this second point of view. The closest example is that of the approach to the evolution of the concept of number as given by Tobias Dantzig in the book Number, the Language of Science.

In the global approach, only a few generic notions are selected to be taught in the elementary grades. The main idea, what is ultimately learned, the basic notion, perhaps inherent in the broadest definition, is taught at a high level of generality consistent with the sophistication of the learner (Bruner, 1960). Mathematicians as well as educators would work closely to determine what is to be taught and to identify the generic notions to be taught. The level of generality would determine the content. In this approach, the use of games to acquire concepts has a logical as well as theoretical place. Games have been used in the analytic approach merely as "aids" or motivational devices, but have not demonstrated the power to facilitate learning of concepts or skills at the level anticipated. Games do not fit into the structure of the analytic approach. In the global approach, the games themselves are the content to be learned. The games serve as the physical models for global or generic ideas. For example, "Mathematical Tic-Tac-Toe" and "Guess My Rule" were first introduced by Robert Davis through the Madison Project to teach the concept of ordered pairs and equations with two variables, respectively. The two concepts, singly and jointly, can be used to teach the concepts of ordered triples, vectors, the coordinate plane, the mapping of functions of various degrees, slope, and the basic notion of derivatives of a function, to name a few.

In the global approach, each idea can be presented through an organizer. Each subtopic is also presented through an organizer. All questions relative to a particular concept or problem are answered in light of the organizer. The "best" organizer is the one with the greatest generality congruent with age and abilities. It should facilitate problem solving and later learning of related concepts. One example of such an organizer is the notion of measurement given in the Mathematics Teacher (Hernandez, 1973). Another is the presentation of the concept of homomorphism (Krause, 1969).

In this approach, the organizer used may be a game or a definition committed to memory. The rules and strategies of the game become the generic notions that are explicitly pointed out by the teacher as related to a particular mathematical idea. In the case of the definition, the teacher uses it as a point of reference for the differentiation and subsequent integration of subnotions. At all times the teacher deliberately and consistently points out the relation of the organizer to the subject under discussion.

If this view of the identification of subject matter is accepted for implementation, the basic skills are the ability to use generic notions as represented by organizers to solve problems and develop additional concepts. It may be that in this approach the teaching of what we have traditionally understood to be basic skills, i.e., use of algorithms for the four operations with whole, positive rational, and positive real numbers, and problem solving involving these operations, will need to be delayed a number of years while the basic notions are developed. If this comes about, it may turn out that students will be better prepared to recognize the need for these skills, and the outcomes may be more congruent with our expectations as educators.

The probability that the suggested approach will be implemented on a large scale is small. What should be recognized by educators charged with the responsibility for making curricular choices is that there is another way to look at the presentation of mathematical ideas that has not been used in the past; that this new approach is significantly different from the traditional method and that it is possible that a large number of students may profit from its use particularly when a combination of the two methods rather than approaches based solely on one or the other is used; that effective teaching of skills may require approaches that are significantly different from those for effective teaching of concepts and that those methods need to be identified. Should such a bilateral approach be taken our task will take on a direction that is significantly different from that followed heretofore.

Suggested Research

It has long been accepted that researchers are influenced by their own point of view on the subject under observation in the identification, selection, description, and implementation of research questions. If this is the case, a number of questions, in addition to the examples suggested below, may be generated by considering a different point of view of the learning and teaching of mathematics. Such considerations may assist researchers in investigating areas that would have remained hidden had a different approach not been suggested.

Several questions come to mind on the teaching of mathematical skills, if viewed from a global approach to instruction. A few are listed below.

1. What is the effect of delaying acquisition of specific skills needed in using algorithms for, say, long division and the operations with rational numbers, in favor of the development of skill in concept acquisition through exploration of global, generic concepts in the early grades, such as operation, number, measurement, on subsequent ability to use algorithms, problem-solving ability, and attitude toward mathematics?

2. What are the processes and problems of language acqusition for both first and second languages, as related to mathematics learning? This is related to topics discussed by Piaget. It is intended, however, that this question focus on the relation of language acquisition to learning mathematics to a depth only suggested by the work of Piaget.

3. What are the interactions among the variables of concept development, language acquisition, and the learning of mathematics? This question is related to 2, and one could possibly subsume the other.

4. What is the effect of learning concepts on the ability to use the concepts to learn skills as dictated by recognized need? This question has been investigated a number of times. It is suggested that additional research be conducted on this question in a format that investigates procedure as well as treatment and results. Refer to question 7 below, for suggestions on a research model.

5. What is the relation of learning concepts to the ability of students to learn related concepts, if the new concepts are approached from a point of view of "having learned to learn"? In other words, will specific instruction on how to learn mathematics, i.e., how to develop new concepts and relations, produce increased ability in learning new concepts?

In addition to asking questions relative to a new point of view of content presentation, there are a number of general considerations pertaining to overall needs.

6. Methods for diagnosis of learning problems in mathematics must be identified. These methods must pertain to learning problems related to language and concept acquisition in addition to skill acquisition in the use of algorithms. The validity and reliability of these methods must be developed to a degree comparable to what has been developed in the field of reading.

7. Models must be developed for the evaluation of mathematics learning. Alternatives to standardized tests must be explored for evaluating student outcomes in relation to mathematics learning. For example, the National Council for Teachers of English have identified alternatives to the standardized test in measuring English skills; these alternatives include folios of student work, interviews, and peer evaluation. Furthermore, the Task Force charged with developing these alternatives has set out a set of 11 criteria for the interpretation and use of standardized test results. Standardized achievement tests were designed to compare groups of students with respect to generally defined knowledge or skills. However, these are used, inappropriately, as measurements of all types of educational achievement. It is suggested that criteria parallel to those developed by the English Task Force be developed for mathematics.

8. Models must be developed for the evaluation of mathematics instruction. Procedures other than descriptive and statistical methods should be utilized in evaluating instruction. The total climate for instruction must be evaluated in a formative manner. Goals and objectives must be identified, evaluated, and modified; methods for implementation of goals and objectives must be identified. evaluated, and modified, as well as a consideration of student outcomes made, as outlined in the suggestion in 7. Valid evaluation of instruction cannot be made when student outcomes, as defined by standardized test scores, are separated from evaluation of preparation for instruction and from the evaluation of classroom activities itself. To do otherwise, the results of no significant difference will continue to be manifest. The "treatment" as defined by research in the traditional manner must be redefined to include many more variables than has been done previously.

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MATHEMATICS EDUCATION

Peter Hilton

Basic Mathematical Skills and Basic Mathematical Education

Before attempting to describe a set of basic mathematical skills or to list the items which, in my judgment, appear in a basic mathematical education, I would like to clarify one point I regard as crucial. When we speak of basic mathematical skills, we are speaking of the basic skills that are used in mathematics. We must not identify skill with mathematics; and we must not confuse skill, which is required in order to be able to do mathematics effectively, with the mathematics itself. I believe that this confusion lies at the heart of some of the difficulties facing mathematics education today. I believe that this confusion has misled many reform programs into concentrating on an attempt to identify and measure skills and have led those movements to neglect the lifeblood of mathematics in the process. I have said enough about this elsewhere (Hilton, 1974, pp. 77-104) for it to be unnecessary to reemphasize the point here. However, I would not wish the point to be overlooked.

For this reason I prefer to talk of basic skills in mathematics and basic mathematical education. With regard to these basic skills, I am very much in agreement with Joseph Rubinstein, who lists a set of such skills in his paper. It is not surprising that we agree because we are colleagues on a program now being developed. Indeed, I will be content simply to emphasize certain points also found in Dr. Rubinstein's paper.

The most basic of all skills is that of competence in the use of number systems for counting, comparing, ordering, and measuring. I would place particular emphasis on the ability to pass easily between the different uses of the number system, and the related ability to use the appropriate model in order to interpret and justify the arithmetical operations. For example, it is clear that in the use of the numbers for measuring, that is, in the representation of the numbers by means of the number line, one obtains the most natural interpretation of the addition operation on negative numbers.

Another important skill is that of being able to handle the various units of measurement appropriate to a given physical situation. This skill must, of course, include the understanding of the means of transferring from one unit to another for measuring the same physical quantity. It should not need to be stressed that all students must be skilled in handling the metric system.

An additional skill must consist explicitly of handling ratios, fractions, and decimals, and the ability to pass among them.

The organization and arrangement of data constitute yet another basic skill; so, too, do the abilities to approximate, estimate, and recognize situations in in which approximation and estimation are the appropriate procedures. However, here we are really leaving the domain of skills and entering that of mathematics itself, as a living science, enabling us to interact effectively and efficiently with the world we live in.

For this same reason, I do not think one should properly include probability and statistics among the basic skills because the relevant skills are those of arithmetic (and, to a small extent at this level, geometry). Topics such as these will now appear as I come to discuss basic mathematical education.

With regard to this topic, I must begin by saying that it is in a sense prior to that of the basic skills, for one must determine the basic skills by reference to what one wishes the students to learn. No skill is, by its own nature, sacrosanct. The purpose for which students should be learning these basic skills is in order to do mathematics effectively. To avoid any possible misunderstanding, I must stress that when I speak of doing mathematics effectively I do not at all mean "doing mathematics for its own sake." To do mathematics effectively means, in this context, to use mathematics in order to inform one's understanding of and control over one's environment.

With this disclaimer, which in view of the purpose of this paper, I would very much hope to be superfluous, let me proceed to list certain essential ingredients of a basic education in mathematics, without restricting myself to any particular stage of that education. First and foremost then, the most basic of all ingredients is understanding the role of mathematical models in enriching our comprehension of the world around us.

Not very far behind this must be the understanding of the power of rational argument and of its domain of validity. The student then must be able to make the best choice of a mathematical model, and, as this implies, he must understand the criteria, often conflicting, that enter into that choice.

The student must also understand the versatility of mathematics. For example, numbers are used for both counting and measuring; fractions can represent numbers or operators; and linear relationships approximate, to a greater or lesser extent, all smooth functional relationships. With regard to this same question of the versatility of mathematics, it is also important to understand how the same mathematical model can represent different physical situations. To give just one of the most famous examples, we may represent any periodic phenomenon by a Fourier series.[1]

Basic to mathematics education is an understanding of the role of probability and statistics. All our inferences from our experience are essentially probabilistic in nature, and the student must be introduced very early in his mathematical education to the idea of statistical regularity. However, it is very important to distinguish between the genuine mathematical theory of probability and statistics and the mere collection of data. In the appropriate use of probability and statistics there must be a scientific hypothesis to be tested. Of course, as we have mentioned previously, the student should learn how to collect, organize, arrange, and present data; but the collection and choice of this data should be motivated by the scientific hypothesis that underlies the inquiry.

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[1] Many of my colleagues at the Conference took the concept "basic" to refer to the elementary level. Most assumed that we were surely confining ourselves to the precollege level. I have supposed that basic ideas in mathematics may be exemplified at any level accessible to the reader.

I have referred to the techniques of approximation and estimation in discussing basic skills. Here, of course, I would include as a basic aspect of mathematics education the ability to recognize situations in which approximation and estimation are the appropriate procedures. This is, clearly, related closely to the fundamental question of the choice of an appropriate mathematical model.

My final ingredient for a basic mathematics education would be an understanding of the processes of applied mathematics. I believe that much nonsense is written and spoken about applied mathematics.[2] However, I do want to emphasize that the mere selection of illustrative examples of a piece of mathematics drawn from outside mathematics does not constitute applying mathematics to the real world. In order to understand how this is done, the student must appreciate the sort of criteria that are used in the selection of a mathematical model, the tradeoffs involved, and must also, of course, have the necessary skill to be able to reason within the mathematical model and make the necessary calculations. The student must also, of course, understand how to check the validity of his theoretical conclusions against the data of the original problem. Also, as part of the student's mathematical education, he or she must be encouraged to derive consequences from the mathematical model that may, in fact, not relate to the original situation that inspired the evocation of the model.

I wish to stress this last concern because probably nobody but a mathematician would stress it. It lies at the very heart of the natural mode of operation of the mathematician. Moreover, it can be justified on the most pragmatic grounds because, as mentioned earlier, the mathematical model is likely to apply to various and diverse physical situations; and additonal conclusions drawn from the model, that might have been extraneous to the original physical situation, may very well be relevant to a different physical situation admitting the same model.

Major Problems Facing Mathematical Education Today

Here again, I would refer to Dr. Rubinstein's paper. I entirely agree that the teacher-training problem lies at the heart of many of our difficulties. The recommendations of the teacher-training panel of CUPM represented an attempt at a realistic set of proposals for the mathematical education of teachers of mathematics. However, it is well known that very few students in their preservice education are able to devote sufficient time to their studies of mathematics to render a program approximating that recommended by CUPM feasible. However, it is essential to stress that the principal ingredient in the preservice education of teachers of mathematics must be mathematics. It is, I believe, necessary to enlarge and spread the role of the specialist at the elementary level, perhaps by developing systems of team-teaching or encouraging the employment in school systems of specialist mathematics supervisors and specialist mathematics master teachers.

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[2] I have attempted a corrective in my article The new emphasis on applied mathematics (Hilton, 1975).

It is common ground that so many of the difficulties experienced at all levels in mathematics education have their root in the extremely unstable and ephemeral nature of the mathematics that children acquire at the elementary level; what is 'learnt' in one year is forgotten the next. At the risk of being boring, I will insist that much of the trouble springs from the overconcentration on skills and memory at the expense of genuine mathematical understanding. In saying this, I do not aline myself with the uncompromising defenders of the new mathematics. However, I will go so far as to say that what the new mathematics was attempting to achieve was, in many respects, very well-intentioned. To the extent that teachers of the new mathematics neglected the basic skills, they were, I believe, misunderstanding the intentions of the pioneers of the new mathematics.

The Role of NIE

Here, I shall speculate and perhaps indulge in Utopian dreams. However, I would like to see NIE become something more than a funding agency, waiting for research proposals to come in that would then be assessed with a view to support or discouragement. I would like to see NIE take a much more positive role in the process of coordinating and even initiating research in mathematical education. I would like to see NIE give some leadership and seek out those able and willing to carry out effective research. For this purpose, NIE should have available a panel of consultants who could give continuing advice as to what were the most urgent and most promising problems to be attacked.

It would also be valuable if NIE could, in some sense, keep a record of what has been achieved. It has always greatly bothered me that in mathematics education, as distinct from mathematics, research does not appear to lead to results that can then be regarded as solid and can form the basis for further research. It seems to me that, in any area of research, the research justifies itself precisely when information acquired through that research achieves the status of dependability and can be used reliably for going further into important questions.

I still find it necessary, in view of my own recent experience, to emphasize strongly that mathematicians should play a prominent role in the counsels of NIE. Of course, I do not mean by this that every mathematician is, ipso facto, worthy of being consulted by NIE on problems of mathematics education. Nor do I mean that no nonmathematicians are worthy of being consulted. Both these views would plainly be nonsensical. What I do mean is that any deliberations with regard to mathematics education should take place with the representation and cooperation of mathematicians who have evinced a continuing concern and feeling for the problems of mathematics education. The lie must be nailed that mathematicians are only interested in the proliferation of their own kind. This may be true of some, but it is certainly not true of mathematicians in general.

Among the particular problems facing mathematics education today are those of teacher education and of teacher-parent contact. I believe that NIE could sponsor programs for investigating ways to improve each of these, and, particularly with regard to teacher education, it may be that quite different modes of operation should be contemplated. For example, it is by no means clear to me that there is any advantage in a student learning, through course work, about methods of teaching before he or she has had some experience in the classroom. Here, I simply name one possible type of experiment. I am, of course, no expert and consequently would not wish, in the circumstances in which I am preparing this paper, to put forward hard and fast proposals. However, I do believe that, with regard to these two urgent problems, as with regard to the whole area of mathematics education, NIE should play a leading role and seek to insure that the best thinking available goes into the attempt to solve some of the most urgent problems facing education and, therefore, society today.

Finally, NIE is in an especially favorable position to coordinate research in education in various fields. It is, I believe, morally certain that good educational procedures in one field (say, mathematics) would influence student performance in other fields. I would like to see NIE sponsor research to estimate the extent of the transfer from one field to another. Surely such transfer constitutes a primary criterion for judging the efficacy of an educational reform.

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BASIC MATHEMATICAL SKILLS AND LEARNING: A RESPONSE

Martin L. Johnson

What Are Basic Skills and Learning?

Webster's Dictionary defines basic as that which "forms the base or essence," or, that which is "essential or fundamental." As applied to mathematics, basic skills and learning will mean those skills and learning that are the base on which all further mathematics knowledge is built.

A look at the literature reveals that much time has been invested into efforts to identify sets of "basic skills and learning." These efforts usually culminate with a list of basic content, a survey of adult perceptions of what mathematics is basic, or, in some cases, textbooks titled or covering "basic mathematics." Ironically, the intersection of the contents of all of these products is small, indicating very little agreement as to what constitutes the set of basic mathematics skills and learning.

Perhaps there is no unique set of basics, or at least no agreed upon set. This, I believe, is as it should be because each composer brings to this problem a unique set of perceptions about self, others, students as learners, and beliefs about what mathematics will be most useful to individuals as they engage in different roles in society.

For example, a mathematician is acutely aware of mathematics needed by the research mathematician and this will probably be reflected in any list of basics constructed by this person. Knowledge of the function concept, knowledge of field properties, and understanding of the concept of variable would be examples of "basic learning" a mathematician might identify. A different set of basics may be submitted by mathematics educators, technicians, housewives, and brickmasons.

While much disagreement remains about what are the basic skills and learning in mathematics, it is still worthwhile to address the question.

It seems to me that many of the lists that have been proposed suggest that the totality of basic skills and learning center around the ability to compute accurately and efficiently with whole number operations. I agree that these skills are important (as will be reflected in a list proposed later); however, the greatest need in our present day society is for individuals who can use their minds as well as their computational abilities. I would propose that basic mathematical knowledge must include the ability to apply mathematical ideas in problem-solving situations. That is, the individual becomes a good problem solver. Unfortunately, there are no neat prescriptions for teaching children to become good problem solvers.

Brownell (1942), Pólya (1945), and others have researched extensively and written about factors affecting problem-solving ability. The research indicates that one necessary factor is to immerse individuals in a problem-solving atmosphere. Opportunities must be provided for individuals to search for patterns and generalizations in sets of data, to perceive interrelationships among seemingly different concepts, and to have experiences that promote a spirit of inquiry at all levels of schooling. Regardless of what is stated as a basic list of content, this content should be developed within the framework of solving problems.

I think that elementary school is the petri dish in which children can be cultured to develop problem-solving abilities. Upon leaving elementary school, each student should display a willingness to tackle new and challenging problems and should have had enough experience and success in this approach to learning to insure little difficulty in problem-solving encounters in later schooling.

There is also a set of mathematics content that I view as basic. Again, all this content can be developed in the elementary school. Table 1 contains an outline of this basic content. Much of the content proposed in Table 1 has been proposed by other writers. There are, however, certain emphases that should be placed on parts of this outline if the intent is to develop better problem solvers.

First, emphasis should be placed on developing some real world meanings for the basic operations. For instance, one might define addition as an operation that is used to find the size of the total in a set whenever the size of the parts is known. Such a meaning (now being used by some mathematics programs) gives the student a way to analyze a problem so that a decision can be made concerning what operation is needed for a solution. We have often taught individuals how to carry out the algorithms. However, have we helped the individual if he or she must be told when to use a specific operation?

Second, while much practice must be provided with the algorithms, the numbers used in the computations should be related to a real world problem as much as possible. Numbers could be generated from trips to the supermarket, measurement activities, or other open-ended laboratory investigations. This type of approach should help students see applications for arithmetic suggested in Table 1.

Many present day educators argue that the basic arithmetic listed in Table 1 is insufficient in today's technical and mathematized society. Although society is becoming more technical and computerized, most people still do not need knowledge of computers. The advent of the hand calculator is hailed by many as the answer to computation problems; but as all who have used a calculator know, the calculator cannot decide which operation to perform. With the advent of the calculator, the ability to estimate the reasonableness of an answer becomes essential, again indicating that computation ability alone is insufficient.

In summary, basic mathematical skills and learning include the list of content in Table 1. Also included is the ability to solve problems using the content in Table 1 in real world contexts. How are we doing in the teaching

Table 1. - Outline of basic content skills

Whole numbers	Nonnegative rational numbers	Geometry	Statistics
Concepts for and relations among	Concepts for and relations among	Identification of basic 2-dimensional shapes and 3-dimensional figures	Constructs graphs to depict data
Numeration System Base ten positional system Place value	Numeration System Base ten positional system Place value	Measurement and computation Length Area Volume	Interpret graphs
Operations: Concepts and meanings Addition Subtraction Multiplication Division	Operations: Concepts and meanings Addition Subtraction Multiplication Division		
Algorithms Addition Subtraction Multiplication Division (through 2-digit divisors)	Algorithms Addition Subtraction Multiplication		

profession in terms of teaching these basics? Teachers report that children are poor problem solvers and that they have much trouble applying mathematical ideas. Standardized tests reveal that in recent years even basic computational ability has declined. What account can be given for these conditions? The next part of this paper addresses this question.

Problems in Teaching the Basics

Although the content outline presented in Table 1 covers a small portion of mathematics presently covered in elementary school, there are still many students leaving elementary and high school deficient in these skills. There appear to be many reasons for this condition.

First, in the initial teaching of the mathematics concepts and algorithms, little time is given to trying to make the concepts meaningful through the use of physical materials. When physical materials are used, there is always a "rush" to get to the symbols and abstractions. In some cases, instruction, even in primary grades, begins on the symbolic level. One first-grade teacher related to me that "I try to stay away from physical materials as much as possible." This is unfortunate, and research has shown that children taught only in a symbolic mode in the primary grades develop inaccurate concepts of mathematics at best. Primary mathematics teachers should have a background in developmental theory with emphasis on how this theory can be applied in the classroom.

Second, teachers do not appear to have the training needed to diagnose mathematical difficulties accurately. For instance, many teachers can determine that a child cannot correctly compute using the addition algorithm involving regrouping. However, the next step is to determine if the difficulty is with the regrouping, with knowledge of basic facts, with the meaning of the addition operation, or with some other prerequisite. To assign more exercises of the same type without some instruction on the cause of difficulty does little to help the child. Few teachers are able to do an indepth diagnosis from which proper remediation procedures can be designed. Teachers also need knowledge about the sequential nature of mathematics as they begin to design remedial programs.

Third, teachers need help with establishing a problem-solving atmosphere for children. In such situations, the role of the teacher changes from one of providing all the answers to one of drawing answers from students through skillful questioning. This approach takes time; that is, it takes longer to develop basic skills through the problem-solving approach than through the traditional "teacher-telling" approach. However, when one considers the alternative, if mathematics is to be made meaningful, the problem-solving approach must be considered. Relationships must be shown between the mathematics in the classroom and the use of mathematics in real world situations.

Most problems related to acquisition of basic mathematics skills and learning identified so far have involved teachers. It is only fair to point out that we are still operating with very little information about how children learn mathematics. Isolated pieces of research seem to indicate that more emphasis should be placed on the developmental level of the child whenever we begin to instruct. Much more information is needed.

NIE Role

NIE can play a major role in obtaining reliable information about how children learn mathematics by making funds available for programs of coordinated studies on selected topics. Consortiums of universities or agencies could be supported to carry out this research. Because many of these studies need to be longitudinal, NIE support should extend for at least 3 to 5 years.

NIE support could also play a major role in establishing centers to train teachers in diagnostic prescriptive procedures in mathematics. The centers would provide a field-based program so teachers could be trained "on the job." Teachers with such training could then begin to effect change in the learning of mathematics by children.

Finally, NIE support in the development of curriculums that evolve from a problem-solving approach is badly needed. Alternatives to the present emphasis must be found, but few individual researchers have the resources to launch studies or projects that will have an impact on present practice. I hope NIE is listening.

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BASIC MATHEMATICAL SKILLS AND LEARNING: THE CSMP VIEWPOINT

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Basic Skills and Learning

The questions we have been asked to address in this paper concern a human activity of great complexity (Moise, 1974) equally important to the learner, the society in which he is an individual, and his role in that society. We have well-defined positions with regard to this activity that are expressed in some detail elsewhere.[1] Indeed, our philosophy of mathematics education is embodied in the CSMP elementary program that speaks more clearly to our positions than any rhetoric we could use to express them. Nonetheless, we will attempt to express our positions within the present context:

1. What are basic mathematical skills and learning?

2. What are the major problems related to children's acquisition of basic mathematical skills and learning?

We restrict our remarks to the learning of mathematics at the elementary school level because that is where our present involvement and experience lie.

The most we can do in response to question 1 is briefly to define our positions: First, with regard to the ingredients we believe necessary to the learning of mathematics, and second, with regard to the role the learning of mathematics plays in the total learning environment of children. Our positions with regard to question 2 will follow as corollaries to these positions.

In the process of responding to question 1, we will take issue with two (not necessarily disjoint) approaches to the learning of mathematics. One proceeds by breaking up the content of elementary mathematics into atomic bits or "skills," partially ordering these, and defining the learning of mathematics as the acquisition or mastery of these "skills" or bits by the attainment of certain levels of competence according to measures of one sort or another. The second proceeds by partitioning the school population by ability and/or deviation from cultural norms and treats the learning of mathematics separately for each element of these partitions.

We take the position that the learning of mathematics is a holistic activity that requires an environment that combines (1) deep didactical insights into the nature of the body of mathematical content, (2) vehicles that carry the content and its applications directly to the learner (and vice versa), and (3) a pedagogy that creates situations in which the growth of mathematical knowledge and understanding is maximized. The result of the proper mix of these three ingredients is a program for all educable children, not "mathematics for the middle 60 percent of the population" or "mathematics for the slow learner" or "mathematics for the upper 20 percent" or "mathematics for the culturally deprived." In such an environment children will acquire those skills that one might identify as essential or basic, but not bit-by-bit in dull destructive sequences suited to no individual, and not in isolation from the necessary enveloping totality of mathematical growth.

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[1] "Response to Managerial Education," Educational Technology, 1973.

It is possible to dissect a living organism to display its component parts, but at some point in the process of dissection the organism loses its essence, that quality we call life. It then does little good to put the pieces back together. The parts of a living organism have a collective function that transcends the organism. This analogy, when applied to the way in which mathematics is much too often presented to children for learning, serves to indicate how strongly we feel that mathematics dissected is mathematics destroyed.

It is clear then that we see the acquisition of mathematical skills as certain threads woven into the fabric of the learning of mathematics. It is a legitimate activity to seek to identify skills related to the learning of mathematics but it is a grave error to identify the process of learning mathematics with the acquisition of skills.

It may also be detectable in what we have said above that we believe the proper presentation of mathematics for learning must take place in an environment suitable for all children. We express ourselves more fully to this point of view. The interactions of children with situations that do not exclude any segment of the population by virtue of socioeconomic level, ability level, race, or cultural background lie at the heart of the proper learning environment. The situations and the vehicles by which the children interact with these situations must be flexible and diverse enough to encompass a wide range of developmental levels in the emotional, psychological, aesthetic, and cognitive realms. It is not a mistake to single out certain segments of the school population, such as the slow learner, the gifted, or the culturally deprived, for special consideration, but it is, in our view, a terrible error to believe that somehow there is a certain piece of mathematics especially suited to each segment, a special pedagogy necessary for each segment, special vehicles for transmission of the content special to each segment. This latter point of view, it seems, leads directly to the deprivation of the children it is intended to alleviate; to ghettos its intent is to dispel.

We now briefly address our position concerning the role that the learning of mathematics should play in the total learning environment of children. In classical and most modern methodological frameworks, mathematics has been presented to children either in isolation from their real life experiences (as well as without any connection with other aspects of their learning environment) or with such concentration on "real life," "concrete," or "culture-bound" experiences that the benefits of the unifying themes of mathematics are lost in the process.

Mathematics occupies a rather special position among the arts and sciences. It is peculiarly close both to the pulse of human experience and the language of the mind.[2] It provides an ideal medium for the interplay between concrete experiences (from real life or fantasy) and abstract structures which are used to analyze and extrapolate from these experiences. Mathematics is not culture-bound nor is it, to any significant degree, language-bound (although you would hardly get this impression from examining most modern elementary mathematics textbooks). Initiation to the learning of mathematics, its applications, and appreciation do not, as in disciplines such as literature, history, philosophy, economics, etc., depend on extensive prerequisite experiences that are not present at the beginning of formal education. For example, the processes of "adding to," "subtracting from," "sharing equally among" are part and parcel of almost every child's preschool life experiences.

Mathematics deals with pattern, classification, relations among objects, structure, iterative processes, each of which plays a central role in life experiences, whether they happen to occur within the generally accepted domain of mathematics or not. But it deals with these universal concepts in such a way that, when viewed properly, its role in the learning environment of children should provide a natural environment for linkages between the concrete and the abstract whether the "concrete" and the "abstract" are fantasy or real life based. Given the proper framework of pedagogy and didactical analysis, what is needed are vehicles that engage children immediately and naturally with the content of mathematics and its applications without cumbersome, clumsy linguistic prerequisites. Such vehicles would provide a creative, open environment for the learning of mathematics, with the potential of influencing the general cognitive development of children as well as stimulating the affective domain of learning. Indeed, it is possible, in the form of powerful nonverbal languages, to provide just such vehicles (Kaufman and Sterling, 1975).

Problems Acquiring Basic Skills

We now address our remarks to question 2. One of the major problems relating to the acquisition of basic mathematical skills and learning is the belief, which in some cases amounts to an obsession, that a definitive answer to the question, "What are basic mathematical skills?" will somehow provide an answer to the question, "What is mathematics learning?" We just do not believe that this leads to the right approach to the learning of mathematics.

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[2] In response to the question as to why mathematics so aptly describes nature, the physicist Heisenberg had this to say:

> On the one hand, mathematics is a study of certain aspects of the human thinking process; on the other hand, when we make ourselves master of a physical situation, we so arrange the data as to conform to the demands of our thinking process. It would seem probable, therefore, that merely in arranging the subject in a form suitable for discussion we have already introduced the mathematics...the mathematics is unavoidably introduced by our treatment, and it is inevitable that mathematical principles appear to rule nature.

The identification of the learning of mathematics with the acquisition of "skills," behaviorally described or not, is destructive to whatever ends the learning of mathematics is applied, from survival training to preparation for higher education. But this extreme stance is just one end of a spectrum of skills-oriented approaches to the learning of mathematics. Also included in this spectrum are most traditional programs and, when you manage to penetrate the linguistic camouflage of modern terminology, most modern commercial programs.

We do not disagree with the need for the acquisition of a great many skills in the process of learning mathematics nor do we disagree with the importance of identifying mathematical skills. One has only to look at the CSMP elementary school program to realize this. We are in basic disagreement with the role many think the acquisition of skills plays in the learning of mathematics. The heavy emphasis on the importance of the identification and the acqusition of skills to the exclusion (conscious or otherwise) of attention to the rest of the learning environment in which these skills are acquired leads to great problems indeed.

The second major problem with regard to the acquisition of basic mathematical skills and learning, in our opinion, stems from the view that the "needs" of different segments of the school population (i.e., the slow learners, the mentally handicapped, the culturally deprived, the gifted) differ so greatly that each requires a different approach to the learning of mathematics; different content, different methods for presenting the content, different situations in which to engage the children with the content.

We believe it is not only absolutely necessary, but possible, to create programs that are suited for different levels of ability and cultural acclimatizations and at the same time provide a consistent framework of content, pedagogy, and vehicles that engage the children with the content. It seems to us that any other approach simply results in the accentuation of the differences among different segments of the school population and, particularly in the cases of the slow learner, the mentally handicapped, and the culturally deprived, cuts them off almost at the beginning from the same possibility of realizing their potential that is available to the rest of the population. Thus, we feel that every effort should be directed toward creating such programs. This is not an easy task. It is tempting to take what seems the easier path and tackle each segment separately, but the long range cost in lost potential and cultural fragmentation far outweighs any possible savings.

We have discussed two very general but basic problems related to the acquisition of basic mathematical skills and learning. The role that we suggest NIE should play in addressing these basic underlying problems is to continue its support of research, development, and evaluation of curriculum projects supporting our positions.

Next, we single out two items for special consideration by NIE for support of research and development of materials for the classroom. The first is potentially a most exciting and powerful pedagogical tool -- the low cost electronic hand calculator. The second is a segment of the school population that has proved somewhat intractable to considerable efforts to improve its lot -- the slow learner.

The Electronic Hand Calculator

The hand calculator is one of the most potent educational tools technology has provided. Its potential positive impact on the learning of mathematics, particularly at the elementary school level, is enormous. This potential is enhanced greatly by the decreasing cost profile and availability of suitable machines. By "suitable machines" for the elementary level we mean those that include only the four basic arithmetic operations, preferably a constant capacity, floating decimal point, and large keys that emit a positive click when engaged and that have sufficient separation between them. We propose three areas we feel are worthy of consideration for experimentation and research.

The Role of the Hand Calculator in Supporting the Learning of Arithmetic. We feel that fears that the free use of hand calculators in the elementary classroom will subvert the learning of basic arithmetic are totally unfounded. Indeed, we believe that exactly the opposite is the case. Far from inhibiting or making the learning of arithmetic obsolete, the hand calculator, when used imaginatively, in the proper learning environment, will enhance and enrich learning in the numerical domain. It will act in a supportive, interactive manner with the growth of children's numerical skills and understanding. It should be as ubiquitous in the classroom as the ruler or the compass.

The Role of the Hand Calculator in Problem-Solving Situations. The hand calculator enables children to engage creatively in problem solving involving estimation, probabilistic situations, and combinatorial situations involving iterative procedures. Further, its use allows children to focus on aspects of problem solving at analysis-of-procedures levels or comparison-of-approaches levels that are not possible when all the necessary numerical calculations must be done by hand.

We do not believe that freeing children from the necessity of calculating by hand in problem-solving situations leads to overreliance on the machine or its use in inappropriate situations. It has been our experience (Carmony, 1974; Papy, 1974a; Papy, 1974b; Hammond, 1974; and Ross, 1974) that when the hand calculator is a natural part of the children's learning environment the students do not use it slavishly in problem-solving situations. They use it selectively, e.g., when they meet a calculation that is beyond their ability, or that is too time-consuming, or when a sequence of calculations within their reach is too tedious.

The Hand Calculator as a "Black Box": The Language and Limitations of the Machine. It is important to dispel the "black box" image of the hand calculator by providing some means to explain the workings of the machine, its language, and its limitations at the children's level. It is equally important to provide experiences for children in which they encounter and explore the limitations of the hand calculator. The following example illustrates this (Hammond, 1974).

Assume that the child knows that $\frac{1}{3} \times 24 = 8$, and that $\frac{1}{3}$ is entered into the machine by pushing, in sequence, $\boxed{1}$ $\boxed{\div}$ $\boxed{3}$. Why does the sequence $\boxed{1}$ $\boxed{\div}$ $\boxed{3}$ $\boxed{X}$ $\boxed{24}$ = produce 7.9999992? (The machine used in this case is the Texas Instruments Model TI-2500.)

In fact, there is a pedagogical tool available that serves the purpose of explaining the hand calculator, the abacus. This tool, when used properly, not only serves as a support for children's numerical development, but, because its language is the language of the hand calculator, can be used for understanding at the level of the child, not only the language and the limitations of the machine, but the way it works as well. The following statement from the "Report of a Site Visit to the Comprehensive School Mathematics Program, May 1975," National Institute of Education, lends some support to this contention:

> [I]n a world of real computers, early familiarity with the basic ideas of computers seems to us well worth striving for, even in this very simple form. We were told...of a second grader who had been playing with one of the little electronic hand calculators now so common. He was asked what he thought was inside it, and he replied, "A little man with a Minicomputer."[3] On reflection, the answer seems to be about as good as one could ever expect from a second grader. It show a clear understanding of the possibilities of mechanizing computation, and if one replaces the little man by an electrical source of power, and the Minicomputer by the various binary circuits, it is not too far from a simplified technical explanation.

The Slow Learner Problem

Our major position with regard to the slow learner problem was presented above. It concerned the point of view from which the problem should be approached: The inclusion of the slow learner in the general progress of the learning of mathematics in the classroom. We take this position in full cognizance, from long experience, of the fact that the slow learner needs special materials and special treatment -- reinforcing materials at all grade levels, materials beyond those written especially for the slow learner, and individual attention when possible.[4] But, at the same time, it must be realized that the slow learner has much the same needs, intellectually and psychologically, as any other child.

Very often a child is labeled a slow learner on the basis of poor achievement in basic arithmetic, whether this is the result of his preschool background, individual developmental factors, or some combination of these. If a program of learning mathematics is based almost entirely on arithmetic, the results are predictable. The slow learner starts behind and progressively lags further and further behind the rest of the class until an unbridgeable chasm develops between the slow learner and the students who progress "normally." This process is exacerbated if the hierarchy of skills involved must be mastered sequentially, which is often the case.

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[3] The Papy Minicomputer is a mixed based (binary, decimal) abacus used as a basic pedagogical tool in the CSMP elementary program.

[4] See "Supplementary Reinforcing Activities for the CSMP First-Grade Program" (St. Louis: CEMREL, 1975).

The slow learner is, in our opinion, tied closely to the nature of the total program in which his or her mathematics is learned. Our specific remarks concerning the slow learner problem reflect this position.

1. Programs constructed using the spiral format (that is, those in which progress does not depend on sequential mastery of skills, but which rather offer continued exposures to content at increasing levels of sophistication) allow for differential rates of growth toward mastery and provide the flexibility necessary for the insertion of special materials into the curriculum for the slow learner without disrupting the flow of the curriculum. Such programs can modify the impact of the difficulty some children have acquiring fluency in basic arithmetic.

2. It has been our experience that in areas of mathematics learning other than basic arithmetic, i.e., situations in geometry, nonnumerical relations, classification problems, elementary probabilistic games, etc., the so-called slow learner can often participate on a more equal footing with the rest of the class. Thus, we feel the incorporation of rich activities and situations in the nonnumerical domains of mathematics into elementary school mathematics programs is not only essential to the learning of mathematics but also offers opportunities for the slow learner to participate more fully in collective learning situations.

3. In the numerical domain the creative use of the hand calculator may have two effects relative to the slow learner problem. First, its proper use can stimulate and support the slow learner's progress in the numerical domain. Second, its proper use should enable the slow learner to participate in interesting numerical situations with the rest of the class.

Finally, we suggest that some consideration be given to probing the relation of the affective domain of learning to the learning of mathematics. The affective domain is not generally associated with the learning of mathematics. CSMP has created a series of storybooks and story workbooks with the purpose of engaging children in mathematical content via situations that we believe have not previously been explored.[5] (The Weird Number, a film produced by Xerox Corporation, is the only exception of which are aware.) We have no definitive results yet regarding the effect such materials have on children and how their learning of mathematics and their views toward mathematics are affected when experiences such as these are integrated into the mathematics curriculum, but we feel there may be certain concomitant results in the effective domain of learning through the use of these storybooks. We suggest this as a fruitful area for research.

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QUESTIONS ABOUT MODELS OF MATHEMATICAL LEARNING

David Klahr

This paper is a response to the two questions that provide the agenda for this conference:

1. What are basic mathematical skills and learning?

2. What are the major problems related to children's acquisition of basic mathematical skills and learning, and what role should the National Institute of Education play in addressing these problems?

I believe that we cannot discuss the second question without first answering question 1, which has two parts: one dealing with what is learned and the other dealing with the learning process as such. I will address most of my comments to the second, and (I believe) hardest part of the first question: What is learning?

My own research is directed toward understanding the fundamental processes that underlie children's developing ability to formulate and utilize processes for representing the quantitative aspects of their environment: elementary quantification (Chi and Klahr, 1975; Klahr, 1973), quantitative comparison (Klahr, 1976b), quantitative classification (Klahr and Wallace, 1972) and quantitative transformation (Klahr and Wallace, 1973).

The relevance of understanding learning processes to the first part of question 1 is simply that the quantification skills being investigated appear to lie at the core of "basic mathematical skills." That bald assertion will not be proven in this brief paper. Rather, I will focus upon the "learning" side of the question, by drawing upon some comments initially prepared for another volume (Klahr, 1976a) in which I attempted to raise some issues that must be faced if we are to investigate directly the process of learning. My hope is that the discussion may suggest fruitful ways to recast the learning question into subquestions that we really can begin to answer with currently available research tools.

What do we know about what happens to a learner during instruction? One of the most exacting criteria for testing our knowledge about any phenomenon is the extent to which we can build a model that exhibits the behavior being studied. If we can simulate it (the behavior), then we have at least a sufficiency model. Of course, many aspects of the model may lack plausibility, but these aspects can then become the focus of further study. (Reitman, 1967, once characterized this simulation approach to cognitive psychology as a way to "invent what you need to know.") In this paper I will raise some questions about how one might go about building a model of a learner in an instructional mode (MOLIM).

Learning as Problem Solving

To learn is to solve a problem. In all but the most elementary situations learning is under the learner's strategic control of attention and memory. If this view of learning is valid, then the study of complex problem solving -- and the orientation such study provides to cognitive psychology -- has direct relevance for the design of a MOLIM. In this section I will mention a few features of problem-solving theory that seem to justify labeling learning as problem solving, and that also have particular importance for the study of learning. In the next section, I raise some questions about the design of a MOLIM.

Current information-processing approaches to the study of human problem solving (e.g., Newell and Simon, 1972) proceed by postulating a general system architecture, and then constructing explicit representations for the data structures and the processes that generate the observed problem-solving behavior. A problem solution consists of an internal representation for some knowledge that the system did not have at the outset. Problem solving consists of a series of local transformations of knowledge that ultimately reach the desired knowledge state. In recent work by Greeno (1974) and Norman, Rumelhart, and Group (1975), the "solution" to the learning problem is explicitly represented as a data structure (semantic net) and a set of procedures for searching that network. But note that these results -- these solutions to the learning problem -- are static with respect to the learning process itself. That is, with respect to the time-grain of the instructional process, the results of instruction, even though they may be dynamic processes, are structures upon which the learning system must operate. We need a model of the system's response to instruction; that is, of its functioning in circumstances in which it must attend to the instructional episode and modify its own performance structures and processes.

In our instructional efforts, we try to provide optimal environments for the human information processing system to learn. As it is with the horse led to water, so it is with the learner in an instructional situation: We can't make it ingest what we offer. The instructional design question is typically "Will the learner learn from this instruction?" A further question should be "Why should he learn?"

The view of learning as problem solving suggests some ways to characterize this question. Problem-solving theory (Newell and Simon, 1972) includes two features important for our purpose: (1) a detailed statement of how the task is internally represented, and (2) a characterization of how the human information processor allocates its limited processing capacity to the problem-solving process. A principal method for effecting this allocation is the use of explicit representations for goals. Goals are symbolic expressions that direct and control the course of problem solving, representing what the system "wants" to do at any moment and "why" it wants to do it. Thus, the answer to whether or why the system will learn becomes, in a view of learning as problem solving, a matter of stating the circumstances under which learning-related goals are generated and manipulated.

Some Design Questions for a MOLIM

The designer of a MOLIM must answer four questions:

1. When should learning occur?
2. How will the system be changed as a result of learning?
3. How thoroughly assimilated is the thing to be learned?
4. How distinct is learning from performance?

These questions and their answers are highly interrelated. Because it is difficult to determine their appropriate order of presentation, ordering here is arbitrary and does not imply any particular differential importance.

When Should Learning Occur?

A curious problem with most of the learning models in both cognitive psychology and artificial intelligence is that they are too single minded in their task: They learn all the time. In designing a plausible MOLIM, we must be able to account for the fact that learning does not occur most of the time. We can do this by explicitly including in our MOLIM the precise conditions under which an instructional episode causes something to be learned. This is where the appropriate use of goals could play a role. Rather than construct a system in which the tendency to learn is integrally built into the underlying operating mechanisms, we can design a more general problem solver whose problem is to learn and whose goals include explicit learning efforts.

By themselves, such goals would still be inadequate for deciding when the system should learn. Additional information would be required about the current state of knowledge -- that is, about both the current configuration of the external environment as well as the internal state of the system. Thus, another design decision concerns those variables and their critical ranges which would, in conjunction with the learning goals, activate self-modification processes.

Mechanisms that determine when learning is to occur must be capable of representing differential responsiveness to instruction. As Resnick (1976) has pointed out, our models must be able to represent both early and late forms of task proficiency; and for a MOLIM, the task is learning itself. Therefore a MOLIM must incorporate the capacity to represent both early and late learning proficiencies. Siegler (1975) has noted the importance of experimental designs in which both older and younger children are given the same training sequences in order to examine the possible interaction of age and instructional effects. Since such interactions have been found (e.g., Siegler and Liebert, 1975), we must be able to represent them in MOLIM through the general strategy, suggested by Resnick, of building developmentally tractable models.

How Will the System Be Changed as a Result of Learning?

This is, perhaps, a more useful way to ask "What is learned?" Several outcomes can result from the learning effort. (1) Nothing happens: The learning attempt fails, and no lasting change is made in the system. As noted previously, this is common in real instructional situations, and we must be able to build a system that can handle failure to learn. (2) The entire system architecture could change: The system's components and their interrelations might be altered. However, since by "system architecture" I mean "hardware" rather than the "software" of the human information processor, it seems unlikely that this kind of change is really the result of instruction. Although a full developmental theory must pay attention to system architecture, we need not be too concerned with it for now.

There are two kinds of software changes that the system can undergo: (1) changes in processes and (2) changes in structures. Newell (1972) has demonstrated the imprecise nature of the process-structure distinction in systems that are themselves undergoing change. In the case of a self-modifying system, the ambiguity becomes even greater, because it is linked to the issue of degree of assimilation.

Although one often makes an apparently unambiguous distinction between instruction directed to acquisition of facts and instruction directed to teaching of procedures (or skills), it is clear from the recent work of Greeno (1974) and Norman (1975) that the issues are not so simple. One can represent "factual" knowledge by procedures that can generate those facts, and, conversely, one can represent what could be procedural outputs by appropriately complex "static" symbolic networks.

Another type of change that may result from instruction and that we must therefore be prepared to represent explicitly in our MOLIM, is change in the learning properties of the system beyond the representation of the specific instructional material. For example, in the case of the aggregate models with which Atkinson (1972) represents the learner, there are few changes in the acquisition parameters that result from the instruction. In more complex models of learning, such systemic modifications would include the basic rules of self-modification.

How Thoroughly Assimilated Is the Thing To Be Learned?

Newell (1972) distinguishes between several levels of general versus specific knowledge about a task. The more general the knowledge, the more transformational rules are necessary to take the system from its entry state at performance time to a task-specific state where it can actually perform the task at hand. Conversely, a very task-specific piece of knowledge might be represented in "machine code": being fully assimilated it would require no interpretation at run time; however, it would be of limited generality.

A concrete example of this distinction is provided by the models for children's performance on seriation tasks developed by Baylor and Gascon (1974). In these models there are two kinds of representations for "seriation knowledge": (1) a base strategy that consists only of series of nested goals that describe, at the most general level, a strategy for seriation (e.g., "find max," or "insertion"); and (2) a rule set that accounts for each move made by the child during a specific seriation task. Behavior during length seriation has one rule set, and behavior during weight seriation has another. If the system has only the base strategy, then it also requires a set of rules that takes the base strategy and constructs a task-specific variant (e.g., for weight seriation). There are various ways to conceptualize this mapping. The two simplest ways are a complete "compilation," in which the base strategy, plus the task-specific mapping rules, create an entire task-specific system that then "runs" on the task. The other is a collection of interpretive rules that never creates a task-specific entity, but instead interprets the base strategy, "on the run," in terms of the specific task.

In designing a MOLIM, we must decide upon the assimilatedness of the information to be acquired. The semantic networks of Norman and Greeno appear to be far toward the task-specific end of the spectrum, while the perceptual theories of Shaw and his colleagues (1974) appear to be focusing upon a more general "base strategy" in their representations of group generators. A similar contrast can be found in comparing Atkinson's (1972) with Resnick's work. Atkinson aims at an instructional procedure that will create a very specific set of data structures and processes that enable the learner to acquire a second language. Resnick (Resnick and Glaser, 1976) has begun to investigate how the learner abandons the task-specific instructions and creates a more efficient and general procedure. My strategic bet is that by representing the result of learning as "base plus interpreter," we may begin to understand the mechanisms of generalization from, or beyond, the specific instructional sequence.

How Distinct Is the Learning System from the Performance System?

In almost all models of learning (psychological models or examples of artificial intelligence), there is a clear distinction between the learning processes and the thing-to-be-learned, i.e., the performance system. (See for example the models in Feigenbaum and Feldman (1963), or Simon and Siklossy (1972).) The distinctions are made with respect to the overall organization of the respective systems, the underlying representation and even the basic system architecture. For example, in the letter series completion model of Simon and Kotovsky (1963), much attention is paid to the differential short term memory demands made by different representations for different serial concepts, but the demands made during the induction of these concepts (i.e., during their learning) are not directly addressed. Another example of this distinction can be found in Waterman's (1970) learning program in which the result of training was represented as a production system, but the learning system itself was not a production system.

Although such separation has the benefit of making the modeling task more manageable, it lacks both elegance and psychological plausibility. I would hazard the guess that the same mechanisms that span the gap between general base strategy and the task-specific system are implicated in the learning process itself. The more homogeneously we design the MOLIM, the more likely we are to be able to solve both problems simultaneously. Such a view might be nothing more than idle speculation were it not for the recent work of my colleague, Don Waterman. He has constructed a set of adaptive production systems for a range of learning tasks (Waterman, 1974a, b). These models learn simple addition, verbal associations, and complex letter series. Each model is written as an initial core of productions, some of which have the capacity to add additional productions to the initial core. The final "learned" system operates under the same control structure and system architecture as the initial system, and the learning rules are represented in precisely the same way as the new rules that are learned, i.e., as productions.

The instructional environments in which Waterman's system do their self-modification are relatively simple, but I believe that the basic approach is very sound, and can be extended to richer instructional problems. In a less precise but much more general statement, Wallace and I (Klahr and Wallace, 1976) have proposed a broad view of cognitive development in terms of a self-modifying production system.

Conclusion

Having posed the design issues, we might ask a few questions about the enterprise *per se*. Why bother with such an effort? There seem to be a few good reasons. First, if we could actually build a sensible model, we could directly simulate the results of proposed instructional procedures. The potential value of such instructional "pilot plants" is that they could replace the extensive fieldtesting of instructional variations that we are presently forced to use. But, except in the most simple situations, we are not yet able to build such models. The worth of the enterprise lies in its propaedeutic nature: It may give us an introduction to the kinds of things we still need to know. But the issue cannot really be addressed in the abstract: We need concrete examples. Thus, the second reason for attempting to raise some design issues is that the exercise of constructing a model of learning from instruction will provide us with such concrete examples.

Another general question that we can ask about the design of a MOLIM is "Who cares?" Who might benefit from such an exercise? It seems premature to claim that either instructors or learners (at least as traditionally conceived) could benefit much from thinking about the design of learning models. The payoff, at present, appears to be for the people who fall into the intersection of the categories of instructional designer and cognitive psychologists. Many of the contributors to this meeting were selected because of just such a blend of interests and skills. Their answers to some of the questions I have raised will be implicit in the work they have presented here. Perhaps other "learning engineers" can, in reaching their own answers, begin to apply and direct the kinds of basic research that are required to further our knowledge of both mathematical skills and how they are learned.

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BASIC MATHEMATICAL SKILLS AND LEARNING

John F. LeBlanc

What Are Basic Mathematical Skills and Learning?

In addressing the question, "What are basic mathematical skills and learning?" I would like to make the following point: Any list of basic mathematical skills and learning must include a set of processes that an adult can call upon in using or functioning with mathematics. Further, these processes should be developed and applied in problem-solving situations so that the student also develops some problem-solving strategies.

There are several reasons why mathematics is taught and learned in school. There are aesthetic reasons; there are cultural reasons. But the most basic reason is a pragmatic one; we want adults to be able to use, to function with mathematics in their lives.

The most critical question then to be asked -- and answered -- in generating a list of basic mathematical skills and learning must be: How is an adult expected to function with mathematics?

The adult is expected to function with mathematics by solving the mathematical problems with which (s)he is confronted. The adult is not only expected to have learned certain mathematical ideas[1] but (s)he is expected to know when and how to apply them. On some occasions an adult may need only simple recall of some facts or a straightforward computation using an algorithm. More often, however, (s)he is faced with a more complicated situation; (s)he must know how to attack a problem, what steps should be followed. In short, the adult must possess several processes or a strategy to solve the nonroutine applications of mathematics.

Let me take a moment to define, at least by examples, what I mean by process and strategy. By a process I refer to a general intellectual function which is applied to a mathematical idea. Perhaps there are some hierarchical levels of processes. At least, it seems clear that some processes are easier to apply than others. The following list will provide some examples of what I mean by processes.

1. Recall some basic facts about mathematics.

2. Select and compute with a standard algorithm.

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[1] Although I am keenly aware that a careful listing of mathematical topics ought to be included in the basic mathematical skills and learning, I will not direct any attention in this paper to that task. Individuals and groups have given substantial attention to this problem. While more attention is needed, I do not believe that the listing of topics is the critical issue.

3. Estimate (a product, the length of a room).
4. Make observations about quantitative data.
5. Look for a pattern.
6. Collect data.
7. Organize information.
8. Compare information.
9. Make interpretations.
10. Make decisions.
11. Evaluate the solution.

While on occasion an adult can solve a problem by applying a single process (computing the total cost of several items) (s)he is likely to be faced with more complicated situations. More complicated situations can often be solved only by employing a strategy. By a problem-solving strategy I mean an organized plan of applying several processes to lead to a solution. The following are some examples of problem-solving strategies:

1. Making a generalization through pattern finding.
2. Using successive approximations (trial and error).
3. Decomposing a problem into simpler subproblems.
4. Simplifying a problem.
5. Recalling a similar problem.

In each strategy one could devise a flowchart of processes[2] that one plans to apply toward the solution of the problem. For example, in a simple problem such as comparing the better buy between two products (9 ozs. X Chips at 59¢ or 10 ozs. Y Chips at 65¢), we hope the adult would have some strategy like: computing unit cost of X Chips; computing unit cost of Y Chips; comparing unit costs; deciding appropriately.

Another simple problem might be to determine the total area (for carpeting, covering with plywood) of an irregular shape. A strategy an adult might use is to decompose the area into more regular areas (triangular or rectangular), to compute each area and to find the total. The problem could be more complicated if the material comes only in certain widths or must be laid in a single direction.

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[2] In addition to the processes and strategies necessary for adults to function with mathematics there are also some skills which might be called tool skills. These tool skills help an adult carry out the process (s)he has selected. For example, constructing a table may help in looking for a pattern; constructing a graph may help to compare data or to make an interpretation. More obvious tool skills are adding and subtracting.

The examples I have given are simple problems. As problems become more complex, so does the flowchart of processes which make up the strategy.

The adult is expected in his personal and career life to solve problems involving mathematical data. If we expect adults to solve problems involving mathematical data then adequate preparation for that task must be given in the schools. Students must learn through using a range of processes and tool skills. Presently, processes such as recalling, selecting, and computing with some standard algorithms are emphasized. There is little or no attention given to the higher order processes. Attention to problem-solving strategies is almost nonexistent.

If an aim of mathematics instruction is to provide adults with the basic skills and learning so that they may use these skills to solve problems in their lives, then direct instruction in a set of processes and strategies seems essential. It would be presumptuous to assume that, without some direct instruction in the processes and strategies, adults would be able to apply mathematical ideas to their problems.

In order for adults to possess these needed basic skills for problem solving, I see two instructional thrusts on students' learning.

1. Beginning at the earliest grades, children should be encouraged to develop higher order processes other than those requiring recalling a fact or selecting and computing with an algorithm. Children can look for patterns in addition tables; they can compare various heights by constructing graphs; they can collect and organize information. Graphic displays of data which children have made have proved to be fruitful ways to have children make decisions concerning mathematical data. Direct instruction on the various processes should precede the direct instruction on p[illegible]solving strategies. On the other hand, children could describe a plan they might use to determine which of two or three packets of school paper would be the best buy.

2. Even before direct instruction on problem-solving strategies occurs, students should be presented with numerous situations involving problem solving. The problem-solving activity should provide opportunities for creative thought and not be readily solved by direct recall or computation. On the one hand, the situations should not be so difficult that children would become totally frustrated. On the other hand, premature formalization of a strategy can also be harmful.

Recently I was with two students, a sixth-grade girl (Michele) and a tenth-grade boy (David). Both students have about the same ability. They saw that I had put some problems on cards with which I had been working. They asked to work one. The problem I gave them was a whimsical one.

> Tom said, "I see pigs and chickens. There are 18 in all." Mary said, "I see 50 legs." How many pigs were there? How many chickens?

In about 30 seconds, Michele said 11 chickens and 7 pigs. David looked aghast and said, "Hey, I just got the equation written down!" Michele had scribbled:

$$\begin{array}{r} 36 \\ \underline{18} \\ 54 \end{array} \qquad\qquad \begin{array}{r} 32 \\ \underline{20} \\ 52 \end{array} \qquad\qquad \begin{array}{r} 28 \\ \underline{22} \\ 50 \end{array}$$

There are several questions which this incident might suggest. Do adults feel mathematics problems can only be solved using formulas and equations? What are we doing to help develop young children's natural problem-solving abilities? Does the current mathematics program not only stifle the problem-solving ability of young children but actually formalize it to a nonoperative point?

In summary, adults function with mathematics by solving problems in their lives. In order to solve problems, adults should have instruction and training in problem solving. There are basic process skills and strategies that are prerequisite for problem solving. I urge that an appropriate set of these process skills be identified and included in the set of basic mathematical skills and learning.

What Are the Major Problems Related to Children's Acquisition of Basic Mathematical Skills and Learning, and What Role Should NIE Play in Addressing These Problems?

In answering this question, I would like to make the following points. The following are among the reasons for the difficulties that students have in learning mathematics:

1. There is a lack of emphasis on mathematical processes and problem solving.

2. The way in which many topics are developed is confusing because of a dual agenda in the instruction.

3. The standardized tests control the placement of many topics.

4. There is inadequate early diagnosis of learning disabilities.

NIE should support the following efforts to improve student acquisition and adult usage of basic mathematical skills and learning:

1. NIE should support research and development efforts that would identify an appropriate hierarchy of process skills in mathematics and an appropriate curriculum to develop these skills in students.

2. NIE should help to establish some clinical centers whose function would be to train specialists in diagnosis and prescriptive instruction for young students who have learning disabilities in mathematics.

There are probably as many reasons for students having difficulty in acquiring basic mathematical skills as there are students who have those difficulties. I have chosen just four of these reasons from among a list of fifteen that I compiled for this task. Within the time and space limitations of this paper, my experiences as a teacher and curriculum developer have led me to focus on the four issues described below.

1. I have already described by implication in answer to the first question what I believe to be the most serious flaw in the school curriculum design: the absence of process skills and problem-solving strategies development. There are some teachers who attempt to help students gain these skills through appropriate questioning and projects. There is presently very little incentive for teachers to promote such skills. The standardized tests do not attempt to measure process skills; there are few, if any curriculum materials designed to promote these skills; there are no accountability measures aimed at problem solving; mathematicians and mathematics educators have not promoted process skills in a uniform and articulate way.

I hope that NIE will take the leadership in helping to correct this most serious flaw.[3]

2. Even if one were not after higher order process skills, there are some serious questions about the present curriculum and the way it is taught. In the development of any of the standard algorithms there appears to be a great amount of peripheral development of ideas broadly categorized as necessary for "complete understanding." For example, the development of the addition and subtraction algorithm is preceded and/or accompanied by dozens of lessons on place value (including seemingly complicated expanded notation) and "important" properties of numbers.

There is a dual agenda: to learn the algorithm, and to understand place value. One has to question whether the coincidental development of these topics is an aid to the learner, or whether it results in confusion. I wonder if a more direct route to algorithmic proficiency with analysis of the underpinnings later might not result in not only more efficient but deeper learning.

There are numerous examples of the dual agenda phenomenon in the elementary school curriculum. This phenomenon must be listed as a possible reason for students' lack of success.

3. The inclusion and placement of many topics seems to be a function of the control that standardized tests hold over the curriculum. For example, many educators think that the development of multiplication and division of rational numbers would be better delayed until grades 7 and 8, but since children in the upper elementary grades will be tested on multiplication and division of fractions, no curriculum developers dare to change the order in which such topics are developed.

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[3] Perhaps in the future the mathematics instruction period in the elementary school might be split into two sections, one emphasizing the development of mathematical ideas with appropriate emphasis on computation and tool skills, and the other emphasizing the development of the process skills and problem solving. The merits of such a dual emphasis can be argued. One could also make the analogy between the two mathematics periods and periods used for composition (or creative writing) and grammar classes. In the problem-solving session teachers should emphasize creativity and resourcefulness rather than accuracy in computation. Here the hand calculator would become a most useful tool.

I have been at several meetings in which the inclusion of topics such as Roman numerals, significant digits, and multiplication early in grade 3, has been decided on the basis that items relating to these topics would appear on the standardized tests. It seems long overdue that a strong effort be exerted to remove the control of the mathematics curriculum from the standardized tests constructors.

Perhaps the hand calculator will provide the mechanism needed to look carefully at the placement and emphasis of mathematical topics. Several educators are promoting an earlier emphasis on decimal notation and a deemphasis on work with fractions. The calculator will most assuredly have an impact on computational instruction. Most important, the hand calculator can take the most tedious task out of problem solving — calculating.

4. I have been told by experts in special education that almost 20 percent of the children in school have a learning disability in mathematics. A high percentage of these children have a serious disability. These disabilities can be identified and, if detected early enough, can be corrected. This early diagnosis and appropriate remediation can only be accomplished if qualified specialists have been employed. Middle school and high school teachers claim it is often too late to help the student because of his(her) habits and general attitude toward mathematics.

I do not know what effect remedial reading teachers or reading specialists have on the reading skills of future adults. But I am envious because everywhere I go I see reading specialists while I see none in mathematics. I think we must examine the benefits of mathematics specialists whose function it would be to provide an early diagnosis and prescription for a struggling child who has some learning disability in mathematics.

The Role of NIE

It is clear that NIE cannot solve all of our problems! In choosing the areas that I believe NIE can have the most significant impact, I have selected efforts that relate to promoting mathematics skills and diagnosing learning disabilities.

NIE can serve as an agent for change by supporting appropriate research and development in the area of the process skills and problem-solving strategies. Perhaps one of the reasons that the process skills in mathematics have not been widely supported is that they are not clearly defined. Individuals, including myself, become rather fuzzy and resort to some handwaving when describing and defining processes and strategies. The following is a partial listing of needed research and development activities related to the process skills and problem-solving strategies:

1. Common vocabulary of well-defined process skills and problem-solving strategies.

2. Research aimed at identifying appropriate levels of process skills.

3. Research to establish an appropriate sequence for development of process skills and problem-solving strategies.

4. Development of instruments that will measure the acquisition of process skills and the problem-solving performance of students.

5. Development of instructional materials focusing on process skills and problem-solving strategies.

If there is general consensus that the acquisition of the process skills and problem-solving strategies is essential for adults to function with mathematics, then NIE should strongly consider a major financial commitment to support the efforts needed to effect that acquisition. I see no greater need in mathematics education.

While the research and development activities related to process skills and problem-solving strategies are an admitted long range effort, there is a need for an immediate action to help a large number of children.

I recommend that NIE support the establishment of a few clinical centers for the training of elementary mathematics specialists. These specialists should be trained in the skills and techniques of diagnosing[4] and prescribing appropriate instruction for children with learning disabilities. Such centers should be clinical in the sense that, in addition to the theoretical concepts of learning disabilities and mathematical pedagogy, the prospective teachers should have the opportunity to actually train with children under careful supervision. In general, the concept of clinical training has been too widely ignored. In particular, teachers who serve as remedial instructors often receive their training on the firing line. It seems that money spent to carefully train specialists capable of diagnosing and correcting disabilities in young children would be both socially and economically healthful.

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[4] There is an urgent need for the development of more sophisticated diagnostic instruments. Presently these instruments simply identify the mathematical deficiencies. They are not organized to aid the diagnostician in assessing the individual's particular disability.

BASIC MATHEMATICAL SKILLS AND LEARNING: SOME COMMENTS ON TWO QUESTIONS

C. M. Lindvall

What Are Basic Mathematical Skills and Learning?

Consideration of this question involves focusing some attention on all components of what is being requested. That which is proposed in response to the question is to be basic; it is to be mathematical; and it is to involve both skills and learning. Something which is basic is prerequisite to, or serves as the basis or starting point for, something else. In answering the above question, we are concerned with the identification of those mathematical skills and those student learnings which provide a basis for permitting a student to move on to further learning. It may, then, be useful to answer the question in terms of two categories: (1) Basic mathematical skills, and (2) basic mathematical learning.

Basic Mathematical Skills

Our concern here is to identify those mathematical skills that must be acquired by students if they are to be prepared for continued progress in the subject. In its work on the development of instructional programs in mathematics, the Learning Research and Development Center[1] has made no effort to develop a new or unique content. Our concern has been for the development of a system of instruction which would permit adaptation to the needs and abilities of the individual learner and which would provide answers to basic questions as to how effective lessons and learning experiences can be designed. Since much of this development activity has been carried out within the context of ongoing public school classroom operations, our specific content has, to a considerable extent, been determined by the curriculums of these schools.

However, in designing our program, we have also studied a variety of other curriculums that are used widely in the schools today, and have attempted to insure that our content is a reflection of what is found in some of the more widely used textbook series. This approach to the identification of the content of a primary grades math program has been important, particularly in view of our focus on problems of teaching and learning as they are encountered in inner-city schools, where it is necessary to "sell" the relevance of the program to school administrators and parents on a relatively continuing basis.

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[1] The suggestions presented in this paper are based on work that has been done at the Learning Research and Development Center and on a number of needs and unanswered questions identified in that work.

A description of what we consider the basic mathematical skills, as reflected in the content of our programs can be outlined in terms of three major types of content: (1) mathematical concepts, (2) fundamental operations, and (3) applications. The examples presented here will be limited to the content of the kindergarten and grades 1 and 2, the grade levels where most of the current work of the writer is centered.

Basic Mathematical Concepts. In beginning their study of arithmetic, it is essential that pupils be introduced to certain foundational concepts and that these concepts be introduced in a sequential order and with a variety of concrete representations.

The nature of these key concepts can be suggested by some examples taken from the content of the primary grades. Prior to the introduction of the concept of number, the beginning student is introduced to the comparison of sets. Sets are compared by pairing each element from one set with an element from the other, and the pupil learns to make the decision that the two sets have the "same number" or that one set has "more" and the other "less."

Although most kindergarten pupils have learned to count before coming to school and can, perhaps, make a comparison of sets by counting, the activity of comparing sets by actual physical pairing adds meaning to counting. This activity also provides a basis for the later introduction of simple number sentences and the equivalence relationship. The student also masters the concept of number, of number names, and "other names" for a number. Of course, this involves work on counting, the counting of physical objects and other types of elements. Prior to being introduced to the decimal system, the student learns to group objects into subsets of the same size and to count the number of such subsets and "the number left over."

Fundamental Operations. The primary grade pupil is introduced to the operation of addition by learning to form the union of the two sets representing the addends, counting the number in this union set, and then writing the number sentence expressing this operation. A similar procedure is used with subtraction. With multiplication and division, pupils form arrays of objects, placing a certain number in a row and forming a specified number of rows. Again, as with the teaching of concepts, it is important that the teaching of the fundamental operations be based on the arranging, ordering, and counting of concrete objects. Also, as pupils acquire some mastery of the operations, even at this beginning level, it is useful to provide them with learning experiences that make them aware of inverse operations and of the extent to which the associative, commutative, and distributive properties apply to the operations. This should not involve the formal naming of the property but only an awareness, at the operating level, of where each applies.

Applications. Mathematics can be made more useful and more meaningful to the primary grade child by placing frequent emphasis upon its practical applications. Of course, these applications must be real to the child. Fortunately, many simple applications can be very real to the child, even though they may seem rather artificial to an adult. For example, using a pairing procedure to determine which of two pupils has more blocks (or counting cubes) may be a rather practical task for a primary grade pupil, even though it appears quite contrived to the adult. Many applications might be those that have a potential use on the playground, in the classroom, or in the home. Stressing applications that are meaningful in the home enhances the chances that the pupils will talk with their parents about their mathematics and gain this type of incidental instruction and some reinforcement concerning the meaningfulness of their schoolwork for everyday life.

Basic Mathematical Learning

As indicated previously, the question being addressed in this section of the paper suggests that attention should be given to the "learning" that is basic to the countinuing study of mathematics. That is, we must be concerned not only with the type of content involved, but also with the type of learning, or type of student mastery, that must be achieved if students are to have an adequate basis for moving on in the curriculum.

Past work on the development of mathematics programs at LRDC has placed an emphasis upon the proper sequencing of instructional objectives and units, and upon pupil mastery of each step in the curriculum as an important criterion for determining when the pupil is ready to move on. We still feel that this provision for mastery learning is an important component of a program that attempts to insure that student learning at one level will provide a sound basis for learning at subsequent levels.

However, it is obvious that the type of mastery that students acquire at each step is a crucial determiner of the extent to which they are prepared for study at the next step. Some of the aspects of this necessary type of mastery, particularly as exhibited in the work of the primary grades, are suggested by the following categories.

The Relationship to Concrete Reality. It is probably unnecessary to emphasize the point that mathematical activities in the primary grades should focus on the manipulation of things, on sorting, classifying, pairing, counting, grouping, regrouping, and many similar activities. Teachers should be relatively slow in moving pupils into work with the symbolic representations of mathematical relationships and operations. They should make sure that students are always capable of representing the symbolic by the concrete. True mastery must involve this latter capability.

Ability to Generalize to Many Representations. The needed understanding of a concept or operations should involve the ability to demonstrate it in a number of settings or representations. A variety of kinds of sets should be compared. Counting should be applied to a number of types of elements. Addition and subtraction should be carried out with a variety of entities. The ability to generalize is essential to the type of mastery required for success at subsequent steps in the curriculum.

Relating Mathematics to Practical Experience. Mastery of mathematics should involve the ability to see it as a useful tool. To the primary grade pupil, reading is a meaningful skill because it permits one to read the comics and other literature important to children of this age. In the same way, mathematics can be made meaningful as it is applied to situations that are important to the child.

Acquiring Interest and Confidence. A crucial aspect of pupil learning of mathematics is the development of an interest in the subject and the development of self-confidence in one's ability to master it. Important elements of preparation for "moving on" are an interest in further study and the conviction that this study can be carried out with continuing success. Experience suggests that the absence of these attitudes represents a major reason for pupil difficulty in making progress in the study of mathematics.

What Are the Major Problems Related to Children's Acquisition of Basic Mathematical Skills and Learning, and What Role Should the National Institute of Education Play in Addressing These Problems?

In response to this second question, this paper defines a general problem area and suggests a number of issues that should be investigated if progress is to be made in that problem area.

It is probably most useful if the "major problems related to children's acquisition of basic mathematical skills and learning" are examined in terms of the problems of particular types of children. In fact, it is likely that there are no problems of major significance that are encountered by pupils of all levels of ability.

The experiences of the past 15 years in the rather widespread introduction of innovative math programs, many with a relatively sophisticated content, indicate that a substantial percent of the upper ability level students really have no problems in their study of mathematics, no matter what the content may be. However, the evidence makes it equally obvious that average and below average students continue to have difficulty. This is particularly true of pupils in inner-city schools, the pupils with whom we have been most concerned in many of our efforts at LRDC.

Results from standardized testing programs indicate that pupils in the schools of many of our large cities are scoring well below national norms. If these tests are accepted as one criterion of success, it seems obvious that one set of "major problems" related to children's learning of mathematics are the problems of teaching average and below average students in inner-city schools. The problems in teaching these pupils are not only those of assuring their mastery of mathematics. Math must be taught in such a way as to generate pupils' interest in math, and in schoolwork in general, and to generate pupils' self-confidence in their abilities as learners.

In teaching average and below average students in inner-city schools, there is a need for particular types of research and development activities centered largely on problems of instruction. NIE could have a unique role in meeting this need. The role of NIE would be to support both research and development activities, to support that research having clear implications for the development of instructional materials and procedures, and to support those development activities that translate research findings into classroom uses. (This suggestion is not intended to have general implications for what NIE would support in the way of either research or development in other program areas. Here we are focusing on ways in which the Institute could contribute to the improvement of children's learning of mathematics.)

The suggestion here is that this program would sponsor research that would lead to relatively specific suggestions for dealing with important problems of classroom instruction. (The nature of this type of research is found in the examples of problems presented below.) This NIE program would also support development activities that involved a carefully planned formative evaluation component and that would produce instructional materials that would eventually be disseminated through regular commercial publishing channels.

Suggested Issues for Research and Development

The foregoing suggestions for a general focus for NIE support of activities designed to improve children's acquisition of mathematics may be clarified by examining a few examples of specific issues. These are presented below, grouped in three general areas. In each case, an issue should be the subject of interrelated research and development efforts so that the results of the overall study would be of the form "This is the answer to the basic question and these are the types of materials and procedures that can be used to apply this in the classroom." (In examining these examples it should be recalled that the focus of this paper is on the problems of teaching in the primary grades.)

Further Clarification of "Basic" Competencies. The earlier discussion of question 1 emphasized the point that it is essential that pupils acquire those abilities and understandings that provide the basis for mastery of higher level abilities. Some examples of questions that might be investigated in trying to determine what is truly "basic" are the following:

1. What are the prerequisite abilities for pupil acquisition of each of the important mathematical concepts and skills? Teachers and other persons (e.g., researchers, developers) who try to assist individual pupils in their study of mathematics are impressed with one fact. Some students have the capabilities that permit them to master a new lesson with a minimum of study and instruction, while others are so lacking in these capabilities that mastery of the new lesson is impossible. But what are these capabilities? What is it that the unprepared student needs to acquire? Further work on the identification of such prerequisites could provide essential information for teachers and curriculum developers.

2. What are the major differences between good and poor students? Important clues as to basic competencies might be obtained through careful studies of the characteristics and abilities of pupils who have little difficulty in mathematics and those who have major problems. What distinguishes the good student from the poor student? If we teach the poor student those abilities or habits possessed by the good student, will the poor student make more effective progress? Answers to these questions could have important implications for instruction.

Adapting Instruction to the Capabilities of the Individual and to the Requirements of the Specific Content. Research and development activities of recent years have resulted in important advances in the area of instructional procedures. But what has been achieved thus far probably only provides a sound basis for further investigations that could lead to answers to critical questions. Some examples of questions needing further investigation are:

1. What are the various types of learning involved in pupil mastery of specific mathematical skills and how can each type of learning be best facilitated? Considerable effort on the part of a large number of psychologists has been devoted to the identification of basic components of learning (e.g., associations, discriminations, generalizations, chains) and to how each type of learning can best be facilitated. Others have attempted to analyze more complex tasks, such as mastery of a mathematical operation, into their basic components. However, much remains to be accomplished in translating findings generated in learning laboratories to their implications for the design of instructional materials and procedures that will be of maximum effectiveness in helping pupils master each of a variety of types of mathematical skills.

2. What are the important individual differences in aptitudes and interests that determine the effectiveness with which pupils can learn mathematics and how can we adapt instruction to such differences? This may well be the most important area for research and development activities that can have a real effect on instruction. All teachers are quite aware of the fact that different pupils have different types of difficulties in their efforts to master mathematics and have been searching for different ways to meet these individual needs. Persons attempting to design programs for individualizing instruction become aware, very quickly, of how little is known about adapting type of instruction to the individual pupil. This is an important area for research and development, one where most of the creative work remains to be done, but also one where there is now some promise that we at least know how to get started with the task.

3. What is the place of drill and practice on "number facts" and computational procedures in the beginning mathematics program? Many current programs suggest that an emphasis on drill and practice tends to make the mastery of mathematics largely a matter of rote memory with an associated decrease in its meaningfulness. It is also suggested that such drill contributes to a distaste for the subject. Others contend that if pupils do not have command of computational skills, they are not in a position to take part in problem-solving activities and in a variety of applications that necessarily require these skills. All of this suggests the importance of research on a variety of problems and of the development of procedures for the use (or elimination) of drill and practice.

Difficulties in Relating the Concrete and the Abstract. One area of difficulty for poorer students in mathematics is that of going from concrete representations of concepts and operations to symbolic representations and of going from symbolic or verbal representations to the concrete. This problem area suggests a number of specific issues that should be investigated. Some examples follow:

1. What are the important purposes and the most effective methods for the use of manipulative activities in beginning arithmetic? The use of materials that the student can count, sort, compare, and manipulate in various other ways represents an essential component

of early instruction in mathematics, but how can such materials be used most effectively? For what kinds of learning are they essential? Must the pupil have personal experiences in manipulating things, or can as much learning take place through observation? Do pupils vary in the extent to which they need to work with manipulatives before they are ready to work with the more abstract symbols for expressing relationships and operations?

2. What are the most effective procedures for introducing paper and pencil activities in association with manipulative lessons? At some relatively early point in their learning, pupils must learn to use the written symbols and sentences that record mathematical relationships and to use simple paper and pencil based algorithms in computation. How can these symbolic activities be introduced and developed in such a way that the pupil can conceive of them as meaningful representations of physical actions and not as somethings requiring rote learning because they are divorced from any real meaning? This is one aspect of the total problem of keeping mathematics meaningful for the student.

3. What are the problems and possibilities associated with the use of language in beginning mathematics? How much verbalization is useful? At what points should the different types of verbalization be required? Can the language of mathematics be simplified in a way which will improve pupil mastery? A significant part of beginning math is the mastery of a number of key concepts. Is mastery of these concepts enhanced by the pupil's acquisition of certain verbalizations, or is it more conducive to progress if initial mastery is at a nonverbal level?

4. What might be the role of hand calculators in early instruction in arithmetic? It has been suggested that future (and perhaps present) availability of hand calculators at a price which would make it feasible for every pupil to have one will have a significant impact on instructional procedures. Will the easy and general availability of such calculators eliminate the need for any emphasis on computational procedures? If this emphasis is decreased, what will be the effect on pupil understanding of basic operations? Can the hand calculator be used to permit the pupil to explore the properties or other features of mathematical operations? Will the use of hand calculators permit the earlier introduction of more interesting problems and applications? Are there other ways in which the hand calculator could be used as an instructional tool?

This list of problems or issues to be investigated is merely intended to suggest some of the types of questions that need to be investigated if instruction in mathematics is to be improved. Many other important problems could be added. The proposal presented in this paper is that one contribution that NIE could make to the improvement of children's learning in mathematics would be to support research and development activities that are directed toward answering questions of this type.

BASIC MATHEMATICAL SKILLS AND LEARNING

Eugene D. Nichols

What Are Basic Mathematical Skills and Learning?

The answer to the question above will vary depending on who answers it. My experience suggests that to elementary school teachers, principals, superintendents, State education department personnel, and parents, the phrase "basic mathematical skills" means to add, subtract, multiply, and divide whole numbers, fractions, and decimals, and to have the most rudimentary knowledge of percent.

Members of the mathematical community usually give a more complex answer to this question. Whether you and I agree with the simple "public" interpretation of basic skills is irrelevant to the issue of what will be emphasized in the elementary school program, and which programs will be the most popular ones in the classroom. The public will choose those programs that appear to satisfy their expectations that children will become good at computing.

Those who interpret basic skills in this simplistic and narrow sense further demand that these skills be taught within the context of their daily use in personal and business affairs. Those who subscribe to the thesis that mathematics should be taught as an organized body of knowledge with its own structure are a very small minority. I count myself in that minority.

To limit the teaching of mathematics to arithmetic in the setting of its immediate usefulness to the practically-minded citizen is to destroy much of the progress of the 20th century. Further, it is to force an early (about middle grades) selection of who shall continue to be educated and whose education shall be terminated.

I prefer the point of view that one of the main goals of education is for an individual to reach a level of intellectual functioning that makes that person comfortably competent in dealing with the demands of modern life. This includes a basic understanding of scientific phenomena. For this, the basic computational skills, as perceived by the groups mentioned above, take a low priority. These are some of the more important skills:

1. Ability to understand data presented in different forms (tables, graphs, etc.).

2. Make intelligent estimates and guesses.

3. Deal with definitions.

4. Ability to comprehend linguistic and symbolic expressions of mathematical entities.

To be sure, the narrowly conceived notion of computational skills comprises a part of these abilities, but emphases and ultimate goals are different -- not skills for the sake of skills, but rather skills for the ultimate goal of intellectual functioning.

There are many careers that make direct use of the most sophisticated aspects of 20th century mathematics; these include probability theory, abstract algebra, and theory of games. It is a fact of life that those individuals who move into these careers do not choose to do so until late in their formal education. Should their early mathematical education be confined to the study of only the basic skills, narrowly conceived, they would be denied entry into these careers. The point is that individuals do not know in advance what knowledge they will need to fulfill their ambitions as they develop. The more knowledge individuals possess, the greater the range of choices of careers open to them, and the greater the potential contribution those individuals can make to society.

The nonmathematical community that is directly involved in education -- e.g., principals and elementary school teachers -- have harbored a quiet uneasiness about the modern mathematics programs that came on the scene during the 1960's. This hidden resistance to the mathematical forces, supported by the scientific demands of the times and the Federal dollar, broke into an almost open revolt in many quarters, during the 1970's. A cool appraisal of what has happened suggests that we overestimated the influence of modern mathematics on the elementary school curriculum. Some 98 percent of the teachers never made any drastic changes in their regular routine. They continued to teach as they always had and waited for the winds of "new math" to blow over. Today the elementary school teacher's main concern is to get those 60 percent of the sudents who cannot compute by the time they get to the seventh grade to be good at computing.

While I am not in sympathy with many aspects of the assessment and write-behavioral-objectives movements (about some aspects I am quite angry), it is necessary to recognize that they have contributed some hard data that we can ill afford to ignore. Let us look at some of the results of the first national assessment of mathematics, in 1972-73 (NAEP, 1975a).

An exercise dealing with multiplication of decimals was done by 43 percent of 13-year olds; 26 percent made an error in placing the decimal point in the answer. This is a level of achievement that is not acceptable after some 7 years of formal study of mathematics.

Here are some other results of this assessment:

1. Fifty-eight percent of the 13-year olds cannot find the answer to $\frac{1}{2} + \frac{1}{3}$.

2. Thirty-four percent of the 17-year olds cannot do the above problem.

3. Thirty-eight percent of the 13-year olds do not know the product $\frac{1}{2} \times \frac{1}{4}$.

4. Thirty percent of the 17-year olds failed to give the correct answer to the question: "If there are 300 calories in nine ounces of a certain food, how many calories are there in a three ounce portion of that food?"

5. Sixty-five percent of the 17-year olds could not correctly answer a question asking for the lowest price per ounce, given the number of ounces and the cost (12 ounces for 40 cents; 14 ounces for 45 cents; 1 pound, 12 ounces for 85 cents; 2 pounds for 99 cents).

6. Fifty-three percent of the 17-year olds could not answer the question posed in the following problem: "A parking lot charges 35 cents for the first hour and 25 cents for each additional hour or fraction of an hour. For a car parked from 10:45 in the morning until 3:05 in the afternoon, how much money should be charged?" (NAEP, 1975b).

I am uneasy about proposing some lofty goals of "mathematical literacy" that encompasses somewhat vague ideas about the basic mathematical understandings, given the harsh data about the majority of students' being incapable of the simplest arithmetic computations. We have no choice but to pay attention to raising the success level of almost all students in these mundane tasks.

I served as one of a three-member NCTM committee on mathematical competencies. Over a 2-year period this committee studied those competencies and published a report in the November 1972 issues of the Arithmetic Teacher and the Mathematics Teacher titled "Mathematical Competencies and Skills Essential for Enlightened Citizens." We found what everyone would expect to find: It is not possible to have universal agreement on a list of specific mathematical competencies that should constitute a minimal requirement for everybody. We did believe, however, that a professional organization concerned with the teaching of mathematics owes the public and the profession a statement that presents a reasonable approximation of competencies prerequisite for a mathematically enlightened citizenship in a modern society. It was in this spirit that this report was published.

Today, I still believe the point of view that we expressed in this report; namely, the role of mathematics can be viewed in essentially three ways (Edwards, Nichols, and Sharpe, 1972, p. 672):

1. Mathematics as a tool for effective citizenship and personal living,
2. Mathematics as a tool for the functioning of the technological world,
3. Mathematics as a system in its own right.

A given adult might be concerned with one or more of these three aspects, depending on his or her career. But the first aspect of mathematics, that as a tool for effective citizenship, concerns all.

To select what specific bits of mathematics should be taught to whom is a very complex task. There is no way to predict, with any acceptable probability, what mathematics a particular individual will need. It is certain, however, that a person ignorant of the most basic mathematics will be automatically denied many job opportunities. This suggests a conclusion that each individual should be given a chance to learn as much mathematics as he or she is capable of learning. This would at least enhance the person's career choices.

Once this conclusion is accepted, it is necessary to take into account the sequential nature of mathematical ideas. Thorough knowledge of arithmetic, algebra, trigonometry, and analytic geometry is necessary for succeeding in calculus as it is presented in current textbooks. And the knowledge of calculus is essential for many professions.

Should it be a matter of free choice, persuasion, or requirement that a given youngster continue the study of mathematics beyond the rudiments of arithmetic? The evidence available from tests shows that, given the normal opportunities to study mathematics in a formal school setting for 7 years, the youngster who has not acquired the basic rudiments of arithmetic is not likely to succeed in further study of mathematics and will not qualify for a job that requires advanced mathematical training. I do not really know what this piece of data means, but it should not be ignored. I would suggest, however, that mathematics should not be required for all students in high school. Should it continue to be a required subject, then we need a kind of mathematics different from the traditional algebra-geometry-algebra sequence. I do not think I like any of the alternatives that have been proposed by various groups so far.

I have no doubt that many of us here could devise, given time and resources, a decent program, from a mathematical point of view. To get the more than 1 million elementary school teachers to teach this program, in a half-decent manner, to more than 30 million elementary school children is quite another matter. The mathematical community has displayed some naiveté about what it takes to introduce a substantial change into the elementary school mathematics classroom. It takes much more than demonstrating that a few selected teachers can get a few selected children excited about a few selected pieces of esoteric mathematics. This could be accomplished at any time in any place. It is quite amateurish to assume, on that basis, that one can drastically change the manner in which mathematics is being taught in a substantial number of schools. The "new mathematics" curriculum of the 1960's did not accomplish this. I agree with Thomas C. O'Brien of Southern Illinois University at Edwardsville, who said that "We have not made any fundamental change in school mathematics" (1973). Furthermore, I am not sure that a truly fundamental change, which would affect a majority of teachers and students, is possible within a short time period.

The preceding discussion suggests that one must consider the question of basic mathematical skills and learning in the context of the total mathematics curriculum. The test results show that we are not doing well. To do better on a significant scale will take a lot of effort and members of the mathematics community cannot expect to do it all.

What Are Problems Related to Skills Acquisition?

In the first section, I cited some examples of what I consider to be an unacceptable level of performance of 13- and 17-year olds on some mundane mathematical tasks. I suggested that we cannot ignore the failure of the majority of students to perform these tasks. Now I wish to put the problems associated with the acquisition of mathematical skills into a broader perspective.

First, I must express a view that it is foolish to expect that everyone should love mathematics. Not everyone should even enjoy it. This is not to say, however, that by means of good teaching and a judicious selection of content to be taught to everyone we cannot increase the likelihood that mathematics can be enjoyed and understood by more people than presently is the case.

While mathematics is a very vital part of modern life, and present civilization cannot exist without it, we should recognize that an individual can get by with very little knowledge of mathematics. Some individuals do quite well without mathematics. But collectively, as a society, we must possess the knowledge of the most sophisticated mathematics and be able to use that knowledge. The foundation for this knowledge must be laid in the early grades, for, unlike literature, mathematics is not usually learned outside the classroom.

While I count myself among the members of the mathematical community, I do not support all that is done in curriculum in the name of this community. We have not taken seriously enough the necessity to understand the elementary school teachers. We saw them accepting and even cheering our pronouncement that our goal was "to produce an American public that is mathematically literate," and we immediately declared ourselves to be in the same camp with them. We have not taken pains to realize that "mathematical literacy" does not mean the same to the two groups. We did not communicate at all. The teachers did not like the embodiment of our notion of "mathematical literacy" that was placed in their hands in the form of teaching materials. Some were shocked by what they were expected to teach.

Further, we did not take seriously enough the necessity of understanding the child. We read a little bit of what Piaget had to say. Because no one really ever understood Piaget, we felt no need to pay much attention to him. But children are rather complex beings and what happens to them mathematically in the first grade is crucial. We members of the mathematical community must take time to understand children as they grow in mathematical thinking. Neither the psychologist nor the early childhood education specialist will do this for us.

What I am suggesting is that the mathematical community, imbued with the powerful knowledge of mathematics, must now learn some new things. Among them is the ability to communicate with the public. I submit to you that such statements as the one following are meaningless to the public. (Braunfeld and Kaufman, 1974.)

> It is an important intellectual discipline (mathematics) that deals with such matters as reasoning, symbolization, abstraction, and generalization. It is a search for patterns in ideas.

It is an interesting quirk of human nature and societal demands that, while the public in the 1960's acquiesced to such statements, the same public rises against them during the 1970's. We have undergone a radical change that I do not claim to understand.

Thus, one of the problems before us is for the mathematical community to learn to understand the teacher and the child. If NIE is in sympathy with this goal, it should support the members of the mathematical community who go to school to learn these things.

I have a feeling that we might be doing a lot of things wrong when teaching mathematics, particularly to young children. We might be trying to teach the wrong things the wrong way and at the wrong time. Research so far provides very few answers as to what should be taught how and at what time. One of the purposes of the Project for the Mathematical Development of Children is to find out what we might be doing wrong and to try to experiment with ways to improve the situation. It is proving to be a very complex undertaking, yet we have hopes that some significant findings will result from the Project's efforts. We started with an assumption that, to be of any significance to education, individuals with knowledge of mathematics must work directly with children and concentrate on interpreting children's behavior as they pursue the learning of concepts and skills in their ordinary school classrooms.

The meager extent to which mathematics is learned by all is part of the total problem of impoverished learning in all areas. There is a widespread lack of faith and value in education on the part of the American people. The parents do not consider the knowledge the child gets by spending each day in school terribly important. This attitude transfers to the child.

NIE's Role

The formation of negative attitudes does not take place overnight; hence, they cannot be changed overnight. Attitudes have a long history of evolution fanned by the mass media, by public officials who make disparaging remarks aimed at educational institutions and personnel, and by lack of financial support for education. NIE, along with other Federal agencies, can help in a vigorous and long-range campaign aimed at reversing this unhealthful influence. Public officials need to be educated in the benefits of education for the quality of the national life.

It is imperative that, as a nation, we invest more in education than we do at the present time. Without that investment, we cannot expect any appreciable improvement in teaching in general, and in the teaching of mathematics in particular. NIE should support more gatherings like this one in Euclid for the purpose of identifying needed areas of development. It should also support efforts to improve mathematical instruction. I am not sure, however, whether we need to develop more curriculums. We already have many good ones, but they are not taught well. We need to find out why it is that so many capable students are not learning mathematics. Ed Begle recently remarked that curricular efforts during the 1960's taught us much about how to teach better mathematics, but very little about how to teach mathematics better.

Many teachers are poor teachers of mathematics because they do not know how to be good teachers. Nobody ever taught them how to be good teachers. We need teacher-training programs that produce good teachers. I believe we can have such programs, but we will not if the responsibility for them continues to rest in colleges of education as they are presently constituted. In too many cases, these colleges frown upon strong requirements for future elementary school teachers to be competent in the rudiments of mathematics. The professors of education who control the training of prospective elementary school teachers consider it unimportant for teachers to be competent in a subject matter. An overgeneralized myth of knowing children in the abstract has been substituted for the study of children's intellectual behavior as they acquire new concepts and skills in their daily routine. I would welcome any effort on the part of NIE in this direction.

In spite of politicians' constant cries that too much is being spent on education, this expenditure is not really so much compared with that for other purposes. It is true that we spend what seems to be a lot for education in grades K-12 during 1974-75 -- $60 billion. But for recreation we spent, in the same period of time, $75 billion. We need to reorder our priorities.

Financial investment alone will not do the job, though. Channeling resources where they would do the most good is necessary. This requires dynamic, judicious, and enlightened leadership, that is presently lacking in public education. Greater support would attract better people, but it would take a long time to effect a noticeable improvement in education. A strong and vigorous campaign is necessary to focus the public's attention on the importance of education. Perhaps NIE can help here.

The teacher's job conditions, particularly at the elementary school level, are extremely unattractive. The elementary school teacher is tied to the children for the entire day -- there is no time to think or to plan -- not even time for a coffee break. Because a profession cannot be any better than its members, and because such poor conditions do not attract the best people, the teaching profession suffers.

Today's market offers the schools and teachers every conceivable kind of teaching material one can wish to have: textbook, workbook, film, filmstrip, tape program, and on and on. But most school do not have money to purchase them. Also, teachers are not trained in the effective use of these materials with the result that, in places where such materials are available, they are not used effectively. More money is needed for the materials and for the training of teachers who will use them.

I have suggested a few specific problems, where I feel NIE might help. One is tempted to suggest that NIE could do something to help solve all the problems I mentioned, but this would be suggesting something that goes beyond the realistic capabilities of NIE. As James J. Gallagher of the University of North Carolina recently wrote:

> There seems little likelihood that the National Institute of Education, with few true friends in the Executive branch or Congress, will be able to mount the effort that is needed to break out of the Catch 22 situation it now finds itself in. That is, it will not get more funds until it proves what it can do, and it cannot prove what it can do without more funds (1975, p. 19).

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NOTICING: AN ASPECT OF SKILL

David Perkins*

This essay full of questions can well start with an anecdote about one. When skills are the puzzle, and mathematical skills in particular, the prodigious facility of the late Norbert Wiener comes to mind. One story has a calculus student approaching the savant with a problem. "Professor Wiener," he asks, "I can't seem to figure out this integral. Can you show me?" Norbert Wiener writes the expression on the blackboard, cocks his head, and after a few seconds jots pi/4 on the blackboard. "But Professor Wiener," the student protests, "I still don't understand how to do it. Can you show me how?" The professor looks up at the ceiling for a moment, and jots another pi/4 beside the first one. "There, you see," he announces, "I did it another way."

Certainly false, this snippet makes all too true a point. The expert's obvious is the student's obscure. Neither readily comprehends the nature of the gap between. Genius as such is not the problem at all. The anecdotal Wiener no more confounds his student than the 10-year old does the 5-year old or the high school senior does the freshman. Take a number; double it; add 20; divide by 2; subtract the number you started with. Your answer is 10. Simply a neat formula for a first year algebraist, this trick can amaze the mere arithmetician. The problem is not level of skill as such, but what divides one level of skill from another.

Knowledge of relevant procedure explains part but not all. If the anecdotal Wiener would describe his derivation. it is likely the student could follow it. He has the knowledge, but fails to foresee its application. Another constituent of skill is fluency with various performances. Where Wiener might carry off a complex manipulation in his head, the student would limp behind with pencil and paper. But here again, given direction, the student would arrive in due course. The parallel problem abides at any level of skill. The student may know the elements of manipulation, be they addition and subtraction or English syntax, quadratic equations or rhetorical devices. How does he, or why can't he, invoke them in appropriate circumstances?

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Skill as Memory

Where difference in skill is the puzzle, chess will serve as well as mathematics. Notorious are the feats of master chess players, including multiple blindfold games, rapid transit, and memorizing boards at a glance. The latter yields crucial evidence: Chess masters can register and replicate a board at a glance only if the arrangement is a normal chess position. For pieces arranged at random, chess masters perform no better than novices. Commencing with this observation, Simon and Chase (1973) assemble several lines of evidence arguing that chess mastery rests, in part, on an enormous vocabulary of configurations of several pieces. With seven or so of these "chunks" (a number that short term memory can accommodate), the master can encode natural board positions. Also, linked to these chunks would be strategic options, conventional responses, and so on. Simon and Chase estimate the vocabulary of configurations a master might possess at about 50,000 chunks, a figure comparable with the English vocabulary of a very literate person.

Such a model reconstrues the casual concept of chess skill. The master no longer appears the superb ratiocinator, deftly reasoning out the soundest tactic on general principles. Rather, his special skill resides in special principles, and many of them. Quick access to so vast a library comes through pattern recognition processes like those that mediate reading or recognition of faces. As the chess master from the novice, Wiener would diverge from his student through owning a large library of strategies and combinations of the basic computational steps and heuristics the student knows, and also by recognizing applicability -- even the applicability of devices the student does know -- in a rapid perceptual fashion.

From the least to the loftiest levels in mathematics, the notion that skill resides in a cumulative body of concepts and procedures seems comfortable. Essentially speaking, any text aims to endow the student with definitions, algorithms, theorems, formulas, problem types, heuristics. A common term for mental structures, "schemes," provides a ready name for all of these without implying any special theoretical affiliation. More psychologically conscious approaches are manifest as well. Skemp (1971) has organized and presented fundamental mathematical concepts as schemas, building on the notions of Piaget.

Approaches to "programed learning" have dissected arithmetic into subordinate, hierarchically organized performances (Gagné, 1963). The LOGO project at the Massachusetts Institute of Technology lays stress on heuristic principles of mathematical thinking rather than the content of particular subjects like algebra (Papert, 1971). Such probings underscore a crucial point: Many relevant schemes remain tacit in the traditional curriculum. Teasing out the hidden schemes clarifies the psychology of performance and works toward a pedagogy insuring that the student is as duly equipped with the usually unrecognized as the usually recognized curriculum.

The work of Simon and Chase warns how challenging this enterprise may be. Though chess lore abounds with strategic concepts -- pins, forks, control of the center, "book games," and so on, those 50,000 chunks outstrip the seeming range of this lore, and hint that the greater part of skill is submerged. The same may be so in mathematics or any discipline. Where it is, any effort to make explicit and inculcate explicity all salient schemes seems hopeless. Rather, education would strive on the one hand to identify the most central covert schemes and inculcate those, and on the other, to explore what conditions facilitate the learner's developing tacit schemes without direct instruction in them.

However manageable these difficulties are, another problem entirely remains. The body of schemes itself can be no more important than activation of those schemes in appropriate circumstances. Where educators have begun to explore what the relevant schemes are, largely ignored has been the recognition process that brings the schemes to bear.

Recognition and Noticing

Compute 3 + 5. Use reductio ad absurdum to demonstrate there are infinitely many primes. Say whether this painting is a Rembrandt. Take the elevator to the fifth floor. Within or outside mathematics, set by others or oneself, such tasks command applications of schemes. Whether the person "has" a scheme is manifested by successful applications. The effort may range from an almost instantaneous act of memory or perception -- adding 3 and 5 or recognizing Rembrandt's style -- to extended matters of riding elevators or reducing to absurdities.

Application is one thing, activation quite another. The above examples express the simplest paradigm for activating schemes. Explicitly given are both the scheme -- addition of digits or reductio ad absurdum -- and the "object" (problem, stimulus, purpose) to which the scheme is to be applied -- 3, 5, or the infinitude of primes. But many problem situations aline little with this paradigm. Often schemes or objects for application of schemes suggest themselves. "Noticing" gives a blanket name to such events, connoting a certain surprise, a discovery not directly sought. "Noticing" becomes a label for the mind's good work in monitoring possibilities beyond or more articulated than the most focal scheme and object.

The term implies a rapid, perceptual act. Certainly a problem solver may arrive at likely schemes via more roundabout routes -- explicit review of known means, elimination of alternatives, and so forth. But with a "big memory" model of advancing skill, rapid retrieval becomes an essential constituent of facile performance -- no matter just of speed, but of success or failure. The function and growth of this "perceptual" dimension to problem solving provides the present focus.

Catching the mind at work in that way calls for a close look at the process of thought. Asking is the simplest tactic. Remarkably detailed accounts derive from chess players or puzzle solvers (Newell and Simon, 1972), artists or poets (Perkins, 1976), and even from children regarding the various strategies they adopt to accomplish arithmetic (Ginsburg, 1974). Noticing is one mental event such reports reveal.

Tracking my own resolutions of some simple word problems yielded several mathematical examples. (The problems were drawn from Wood, 1968.) "At 4:24 p.m., how many degrees has the hour hand moved from its position at noon?" Immediately the words evoked the scheme of proportionality -- the hour hand moving proportionally more slowly than the minute hand. Applying the scheme required fitting the proportionality relationship to the specifics of the problem. A more general "linearity" scheme also applied -- one could treat the 4 hours and the 24 minutes separately and add the results.

In working the arithmetic, opportunities to factor, cancel, and simplify the arithmetic before doing computations surfaced spontaneously. "How thick are the pages in a book?" The question immediately invited a scheme whereby wholes are divided into equal parts, each part spanning the equivalent fraction of the whole. "Determine the diameter of the moon." A matter of measurement at a distance, the problem prompted a scheme of triangulation, which, applied several times in different ways, finally assembled a likely plan.

These examples simply show how noticing can serve such problems. However, the student familiar with proportions might not perceive their bearing on the clock problem. The student who has practice factoring and canceling may not notice opportunities outside the page of exercises. Triangulation may remain a matter for the diagrams in textbooks. And so on.

Lists of basic arithmetical skills from addition to volumes of solids do an important service in defining minimal competencies (Edwards, Nichols, and Sharpe, 1972; Long and Herr, 1973). But all hope for their practical use disperses unless students can bring their skills to bear other than when a teacher or textbook invites them to do so. This realization recommends both a systematic way of discussing how flexibly schemes are activated and a probe of how such recognition skills are acquired. The next section attempts something in the former direction.

Paradigms of Noticing

In the explicit paradigm, attention converges both on the scheme and its possible objects, asking whether the scheme might apply. Noticing can diverge from this in attention to the object. Rather than focal attention, the merest glance or most passing thought may bring one up short -- and initiate application of some scheme. The obvious example is scanning objects with a scheme in mind -- an index for key words, a crowd for a familiar face, a table for a value. In these contexts, as in psychological experiments on perceptual search, attention rests passingly on each item with little apparent registering of information; yet target items are found, suddenly standing out against the amorphous and perceptually unarticulated background of the others. The mind's activation of schemes "on the fly" becomes one significant paradigm of noticing -- here named the glancing paradigm.

Noticing also varies with attention to the scheme. In the explicit paradigm, the person commences with the scheme. But often a scheme may be sought under a more general label. The skilled poet may ask "What flaws does this line have?" and expect to find a forced rhyme or faulty rhythm without seeking them as such. Likewise, the mathematician may ask "How can I solve these simultaneous equations?" and expect a likely approach to suggest itself without an explicit review of his repertoire. Performance in this paradigm might be called looking under. The more general the label a person can look under the more facile his approach to the problem.

Scheme activation occurs yet further removed from the explicit paradigm. Transient or permanent states of alertness may result from extended involvement with various sorts of problems. Opportunities for scheme application may disclose themselves even when not sought as such or under more general labels. The scheme is simple emergent. Sometimes in these cases, activation and application become the same act. To hear one's name called is at once to alert and apply the name recognition scheme.

Mixed cases of glancing and looking under or emergence also occur. Students who leaf through the chapter, hoping for assistance with a problem at the end, are scanning quickly without attending closely, and seeking a particular tactic under an exceptionally vague and general label -- "something helpful." They trust their minds to do the covert work of making a useful match. Perhaps they will. Even "out-of-the-blue" moments of insight may fall under this rubric. A quirk of phrasing, a passing event on the street, may trigger some approach resolving an old problem. Not sought at all, the scheme is emergent. Such happenings attest to the silent alertness that may remain while we go about other business.

The paradigms give a simple way to classify cases of scheme activation, but the examples offered have reserved an important point. The classification depends on previously establishing what counts as the scheme and what as the object -- a matter left tacit above. The reasons are two. First, almost any mental operation can be called a scheme; thus schemes become hierarchical, built on subschemes. Classification of a given event depends on choosing the level of hierarchy. Second, "object" cannot always count simply as the thing perceptually encountered. We notice properties of our thoughts as well as properties of the world. And some parts of the world may be processed before others. Nothing instrinsic declares the boundaries of the "object;" they need to be chosen.

An example will put this in focus. Imagine a car ad whose caption boasts, "The Automobile You'll Recognize!" Readers observe that it is a Cadillac. If the entire ad is the object, and the readers presumed to have no special intent, their recognition classifies as emergent -- entirely prompted by the object. If the caption is partitioned from picture and happens to invoke the car recognition scheme in advance, the event classifies into the explicit paradigm. If there the target scheme is taken to be Cadillac recognition -- a subscheme of car recognition along with Plymouth recognition, Volkswagen recognition, and so on -- the event classifies as a variety of "looking under."

With all this, the paradigms seem a bit of a shell game. In fact, the meaningful classifications ensue when scheme and object are held constant. Suppose the scheme is Cadillac recognition, the objects Cadillacs or their pictures. Then can our readers apply the scheme at all -- can they recognize a Cadillac when explicitly asked about one? Can they look under the general label "automobile brands" and identify Cadillacs? When they see one on the street, do they emergently register "Cadillac" as the ad would hope? Can they scan the parking lot for Cadillacs?

So, when object and scheme are held constant, the paradigms define just how readily a scheme may be activated. Far from a nuisance, the ambiguity of object and scheme yields an advantage. The paradigms become flexible in their placement, adaptive to highlighting what comparisons seem useful and to probing coarser or finer levels of detail. The paradigms function as a coordinate system -- chosen to serve the problem at hand.

Models of Mind

Before exploring some applications, a brief note on theoretical alliances seems apt. In noticing, the object provokes activation of the scheme in a stimulus-response fashion. But this alone marks no behaviorist stance. These pages have unabashedly posited all sorts of mental entities.

The position lies more with cognitive psychology and information processing. A relevant theory of human problem solving is Newell and Simon's notion of "production systems" (Newell and Simon, 1972). They propose that thought can be effectively modeled through a hierarchical structure of statements with the general form "If this condition holds, then do this," arranged by priority of testing and execution. Though the many details deserve a closer look, clearly noticing would fit well in such a system. However, the present discussion does not presume this theoretical backdrop.

Noticing falls outside the province of pattern recognition narrowly construed. There, the abiding issue is extension of recognition to new instances, which may be systematically transformed or obscured by noise relative to the familiar instances (see, for example, Kolers and Perkins, 1975). Under the rubric of noticing, attention shifts from new instances per se to new modes of encountering instances -- glancing rather than direct confrontation, applications searched for under umbrella labels rather than individually, opportunities emergent rather than sought.

Noticing remains an act of pattern recognition broadly construed. At one time or another, research in pattern recognition has touched on all the above phenomena. The present aim is not to review them as such, but to underscore their importance as dimensions of flexibility describing a problem solver's potential in a problem context.

Noticing More

The sharpened notion of noticing leaves unresolved some crucial empirical and pedagogical issues: Who notices more and who less, and how does one come to notice more? These pages can do little more than expand the questions. The obvious makes a beginning. In arithmetic, everyone quickly learns to recognize reflexively the plus sign (+). Whether they can carry through the operation or not, invoking the scheme such as they have it presents no problem. In this sense, the scheme is emergent.

Of course, this marks a change in "coordinate systems." Earlier, 3 + 5 illustrated the explicit paradigm -- tacitly, 3 and 5 were the objects, + declaring the operation scheme in advance. Here transformed with 3 + 5 as the object, the problem fits the paradigm of emergence. The expression as a whole completely declares itself, prompting the addition scheme.

The shift has a purpose. In problem-solving contexts, the natural choice for object is the problem statement itself. The plan is to build a contrast between simple arithmetic problems with their emergent schemes and problems that challenge more the capacities of noticing. One natural priority becomes charting that challenge as it actually occurs. Given problem statements drawing on schemes that students know, what schemes do they notice at once? What schemes to they invoke by looking under? What schemes do they reach by more extended procedures of reasoning or reviewing their repertoires? What schemes do they never reach? What are the individual differences? Process tracing techniques could yield the needed data, and the paradigms give a framework for asking the questions.

Whatever the difficulties of noticing, the reasons for those difficulties require surveying. How might problem statements contrast with "+" to point so much less definitely to schemes? First, the plus sign functions unambiguously to declare the scheme. More precisely, the operation of addition factors into several subschemes, depending on the operands. The plus sign does not uniquely select the subscheme, which depends on the context of digits, integers, fractions, decimal numbers, algebraic expressions, or whatever. The map from sign to subscheme becomes one-to-many, but determinate in context. This and other properties show that the signs of arithmetic relate to schemes in a "notational" way (Goodman, 1968). In particular, sheme determines symbol and symbol in context unambiguously selects scheme.

Nothing in word problems clearly marks the method. Several schemes may solve a given problem, and several more may provide hopeful but awkward or hopeless approaches -- especially as problems grow more challenging. Even the "clock hands" problem mentioned earlier offers an example. A colleague became momentarily confused because the problem statement elicited a "measurement conversion" scheme: Convert minutes to hours or hours to minutes. That scheme leads away from a direct approach that simply considers the hands traveling full circle at rates in constant proportion.

We would also like to think that the problems in elementary texts yield only to the schemes intrinsic to them. But Riedesel (1973) quotes a 1925 description of Stevenson of an elementary school pupil's tactic. "If there are lots of numbers, I add. If there are only two numbers with lots of parts, I subtract. But if there are just two numbers and one littler than the other, it is hard. I divide if they come out even, but if they don't I multiply." As Reidesel remarks, one more nuance -- when in doubt, use the operation most recently studied -- would make the scheme quite powerful.

So problems are not so determinate as they seem. Worse, some solution schemes may pivot on features little connected to the problem -- e.g., artifacts of problem grouping and textbook style. In summary, the ambiguity issue comes down to this: Irrelevant schemes are supported by artifacts of educational practice (the pupil's strategy). Other such artifacts disclose the appropriate schemes independent of the problem statement, reducing the task to the explicit paradigm (grouping problems of one kind together under a revealing title -- the ambiguity problem solved by default). Finally, the problem statement itself may prompt different schemes (the clock example).

The first of these results from minimizing arithmetical difficulties so that students can concentrate on the problem as such -- a good intention needing more cautious practice. The second reflects a stress on scheme application, which surely must be mastered. But such a stress offers little practice in noticing -- in scheme activation based on the problem statement itself. The third points to the real issue of ambiguity, an issue hardly addressed by most learning formats: The more directive the problems within the boundaries of their own statements, the more clearly they exemplify the link from problem to scheme, but the less they exercise detecting such links. How then is noticing best built? Would some kind of mix or sequencing from more to less directive problems best serve?

Beside being unambiguous, arithmetic problems make a certain promise with their +. The scheme elicited surely applies if carried out correctly. Sometimes, as with digit sums, the application is virtually instantaneous; we hope 3 + 5 will yield 8 as neatly as a gumball machine answers a penny. But many schemes in word problem contexts promise neither success nor speed. Schemes are often heuristic in nature, more or less hopeful rather than right or wrong, and sometimes the likely schemes fail and the unlikely ones succeed. Again, schemes may do little more than sketch a plan of attack and, as in looking under, invoke new patterns of noticing to carry the problem forward. For instance, induction may seem a likely approach to a proof. But such a strategy branches into several subschemes. Does a person (1) assume the nth case and prove for n + 1; (2) posit an nth case and seek reduction to any case less than the nth; (3) treat the n-odd and n-even cases separately; or (4) assume a least integer for which the proposition fails or a largest one for which it succeeds, and derive a contradiction -- one way to justify the method of induction itself, narrowly taken?

In sum, reinforcement for noticing a scheme is often delayed and inconsistent. What would be the influence of training that asked students to suggest relevant schemes and offered immediate feedback on their aptness? Would this build noticing capacities? Would it yield an unhealthy dependence on quick answers? Are the students who better develop noticing the ones able and stubborn enough to push schemes through to the end? How do single dramatic successes or failures of a scheme affect noticing?

Another matter of delay arises as well. Where link from problem to scheme is explored here as a rapid perceptual act, such insights may develop out of chains of hunches or reasoning from problem to scheme. With experience, the intermediate steps might drop away. But the role of experience here poses a puzzle. Children and even adults often employ roundabout arithmetic strategies, as 3 + 5 is 4, 5, 6, 7, 8. One might suppose that drill would build a reflexive link of digits to sum. But testing just this, Brownell and Chazal (1935) found that drill simply reinforced the roundabout methods. Does this always happen? When does it happen? Could practice in noticing simply lead students to develop elaborate tactics that never shrink to acts of perception?

Once the + sign elicits the addition scheme, its role is done. The computation proceeds with numbers. But more typically, the features of problem statements that invite the application of schemes are features that figure in the schemes' applications. Recall again the clock problem: A key scheme there was proportionality. Application pursues the characteristics that evoked application: The hour hand moves one-twelfth the rate of the minute hand, and so on. Noticing here rests not in an ancillary symbol like +, but in the first hints of fit between problem and scheme itself.

Does extensive experience with applying schemes of itself develop noticing, by exercising over and over the fit between scheme and problem? If so, concentration on scheme application would let noticing take care of itself. Drill would yield insight -- an unlikely suggestion, but dangerous to wholly ignore. More plausibly, practice in scheme application might prepare the patterns to be noticed, so that the person could be nudged into noticing through a few examples. In any case, the relations between application and activation demand a close look.

Finally, and most blatantly, we know the signal for the addition scheme as such. But precisely what in the clock problem evokes the proportionality scheme? At best, one begins to speculate a bit, and again a cascade of questions follows. In general, what sorts of featuers do figure in noticing? Some analysis of caricature and face recognition (Perkins, 1975) suggests that such patterns may depend on relatively few features of a rather either-or nature, at least for particular schemes. What promise lies in discovering the features for particular schemes, and in teaching those along with scheme application? Could caricatured types of problems, emphasizing the salient cues, better convey them? What if that which seems to make a scheme a good bet in particular cases was simply openly discussed?

All this has been placed in a context of set problems. Of course, such occur not only in textbooks but as subproblems we set outselves in pursuing larger issues. But neither mathematical nor any other kind of inquiry finds its sole foundation in given problems. Like solutions, problems are discovered. And very often, they are discovered by noticing. Certainly if practice in noticing is what strengthens noticing, then discovery of problems, largely ignored as it is, seems unlikely to develop of itself. And here too, all the issues of noticing apply. Most simply, what are the patterns that signal new problems? Possibly of an extremely general nature, if made explicit they might teach powerful lessons. What labels can one look under to find new problems? Brown and Walter (1970) propose one such label. "What-if-not," and a process of generating new problems by questioning the conditions of old conclusions. The same authors note (1972) that listing and examining instances of generalization can elicit many new questions. In effect, the arrayed instances create an object for noticing. The LOGO project at MIT stresses youngsters' undertaking, rather than narrow problems, projects that lead them to discover worlds of subproblems (Papert, 1971). Such explorations of problem discovery as a curriculum element surely cannot come too soon.

Heuristics of Noticing

The above takes better noticing as a matter involving years. But to stop at that oversimplifies. First, problem solvers can better fit their behavior to their noticing capacities. Where general open-ended questions like "How can I approach this problem?" fail, more focused questions may serve -- e.g., "Might induction work?" Where glancing over the problem statement yields nothing, a closer analytical reading may. All too often problem solvers may quit because no solution suggests itself: One can overdepend on underdeveloped noticing capacities. A shift of paradigms toward the explicit offers a remedy.

Other shifts can serve as well. Consider such bits of advice as "Work on a related problem, make the problem more general or more specific, turn the problem upside down, enter the problem in a different place" (Pólya, 1957; DeBono, 1970). All of these are designed to either transform the problem or one's view of it, and thus create new conditions for noticing. This, perhaps more than any other reason, is why such broad heuristics of problem solving and creative thinking actually work a bit. However, like other heuristics, these ideas themselves are schemes -- schemes for noticing. Like other schemes, they must be activated. Schemes for noticing have no special call on being noticed! Offered as a few moments of good advice, they accomplish little. Approached with the same seriousness as other parts of the curriculum, and in the context of how they work, they promise a traditional leverage of mathematics generality.

Such tactics accept and adapt to noticing capacities as the organism possesses them. But noticing, no sole matter of extended learning, shifts with involvement and intention. Commonly, people speak of "getting into" a problem, and research on problem solving endlessly stresses how insights occur after extensive orientation. Involvement appears to generate local and transient patterns of noticing -- perhaps a selective tuning of one's existent noticing capacities.

Important questions follow. Does such orientation just happen, or are there better and worse ways to achieve it? What conscious control can be exercised over such tuning? Strolling down the street, one recognizes only the familiar faces. But ask yourself to classify everyone you meet as a familiar person and the scene will fill with people "resembling a bit" Bob Hope, Bertrand Russell, or your next door neighbor. We seem to have some access to the tolerance levels exercised in recognition, and can loosen or tighten them at will. Does such a phenomenon have any good use? How many like phenomena remain to be -- yes -- noticed?

A Word for Venturing

These pages have played somewhat freely with their favorite term. It is very odd to say someone notices that 3 + 5 commands a sum. Yet, where the case was classified as emergent, this was just the implication. The usage, of course, was "technical," according with the paradigms. That could be the end of it, but the difference in usage has a point to make.

The differance rests in a nuance of everyday speech. One cannot notice the blatantly obvious. Compare "He noticed an enormous rattlesnake hiding under a rock," with "He turned over a rock and noticed an enormous rattlesnake." The latter sounds odd, because noticing always holds out the possibility of not having noticed. One can notice the fairly obvious -- say a strange car in front of one's house -- but not the unavoidably obvious.

In arithmetic, + speaks so clearly that noticing ill applies. The same may occur at the heights of expertise. Where recognition is perfectly sure, noticing becomes out of place. Norbert Wiener notices how to approach the integral less than he sees how to approach it -- easy as 3 + 5.

In one sense, this places noticing as a precursor to very skilled performance, where noticing gives way to simply seeing. But skill in a limited domain like arithmetic and skill in mathematics are two different matters. With the one, once fundamentals are mastered, better performance mostly reduces to smoother, swifter, more accurate execution of the same procedures. The other, larger domain is what Israel Scheffler has called a critical skill (1965). Such enterprise is intrinsically open-ended, irreducible to a reliable set of reflexes.

Whatever becomes a mere matter of recognition, more always remains -- the horizon retreats as one approaches.

In critical skills, one can little afford to depend only on the obvious. No mere heralds to mastery, the most tentative noticing capacities become essential constituents. Noticing gives a name to the mind's persistent excursions beyond the boundaries of the perfectly reliable. Such excursions are clearly essential if new terrain would be won.

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WHAT ARE THE BASIC SKILLS OF MATHEMATICS?

Gerald R. Rising

Did you ever try to reconstitute the egg from an eggnog? That is the kind of task we face in responding to the question: What are the basic skills of mathematics? Both are exercises in futility. Let the egg remain homogenized; we must still address the question of skills. Why? Because major resource allocations are tied to it.

Now that sounds as though I am somehow selling out. Well, those who know my record of negotiation with the Federal Government would tell you that would be ill-advised. The only good thing I ever got from the government is my wife, and she came from a different agency. My motivation here is quite different. As is so often the case when money is involved, the entrepreneurs are already in the wings.

I am convinced that the usual simplistic answers to this question will lead first to development programs skewed to the basic concerns, and second to further misdirection of the energies of our young people. So it is from a defensive stance that I offer the answer that is the heart of this paper. I will attempt, first to mould the question into what I believe we face today, and second, to place the question in its proper mathematical and pedagogical contexts. That indirect response will constitute my answer.

Variables in Basic Skills

It should be evident that any basic skill is a multivariate function. Consider, for example, the following independent variables:

1. Time -- basic skills of 50 years ago are not those of today, and today's skills will probably not be those of a decade in the future.

2. Place -- rural and urban skills differ.

3. Age -- school age skills or pseudo-skills[1] are quite different from the skills necessary to function as an adult.

4. Occupation -- the skills appropriate to a housewife are quite different from those necessary to a medical technician.

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[1] A pseudo-skill is a skill directed at test passing rather than real world application. A job entrance exam may or may not be a measure of pseudo-skills.

These and other variables operate on basic skills in such a way as to make them most elusive. We find ourselves faced with two alternatives. We may seek out only a common ground, the intersection of the skills necessary to each of life's activities; or we may take the union of the same skills.

In each case the result is predictable. In the first, the empty set; in the second, the universe of mathematical -- and quite possibly scientific -- discourse; neither is at all satisfying. Surely there are mathematical skills that go beyond counting that we can designate as basic. Equally certainly, we should be able to delimit our range to some reasonable core. I seek in this paper the means by which we will accomplish exactly this.

Identifying the Clientele. Let us focus the question by designating our clientele. Basic skills for whom? There is a body of skills appropriate to a professional mathematician. We do not seek those here. Rather I interpret "basic" in the sense that it is used in courses like basic mathematics and basic electricity. "Basic" then implies both minimum and central. Basic mathematical skills are those mathematics-related abilities that should be attained in order to function as a citizen. By "citizen," I refer not only to civics attributes but also to career implications. Basic skills should equip people to function adequately in the world of work; they should provide them with a foundation for entry into and functioning within that world.

This last would seem to imply that I accept the union definition of skills. That is not my intention. Rather, I stress the word "foundation": I do not include here the specific skills related to a particular job. I believe in on-the-job training for specific skill acquisition. In fact, I believe that it is exactly this failure to perceive the role of on-the-job training that has thrown much of education out of gear.

To illustrate, let us consider two students: a graduate of Harvard Law School and a graduate of Law School X.[2] The first student has a strong theoretical foundation in understanding of law, but must take a cram course to pass the State bar examinations. The latter has been trained in the specific codes; with schooling directed toward those very exams. Who would you hire? My answer is the same as that of the prestigious law firms: the Harvard graduate. He or she can be given the specific training inhouse and will overcome the initial disadvantage quickly, and have the power to go much further.

My second example is closer to our subject. When I was mathematics coordinator in Norwalk, Connecticut, the senior high school guidance counselors told me of complaints they had received from the Norwalk Hospital nursing supervisors. Our high school graduates entering nursing training there were unable to pass the entrance mathematics test. Despite the fact that these students had studied 2 or 3 years of high school academic mathematics to meet nursing school requirements, the guidance staff proposed to schedule them into 12th grade general mathematics classes in order to remedy this deficiency. These bright youngsters were to be enrolled in classes with the weakest and worst students in town in order to prepare them for this entrance test.

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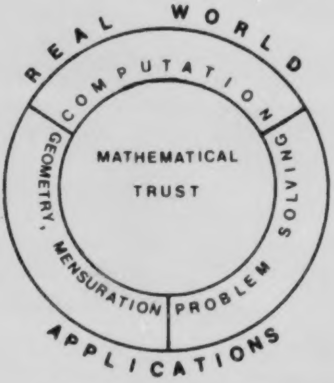
[2] You supply the domain. My X has since been incorporated into a major state university. I hope the designation there no longer applies.

I rebelled at the notion. I checked with the hospital. The nursing supervisor showed me the exam -- all proportional mensuration problems. More important, she told me how her staff corrected the deficiencies: They provided 2 hours of classroom instruction and then readministered the test. Every student passed this second exam. As a result, we offered a similar 2-hour cram course and solved our problem, too.

But why? Surely these problems are job specific and belong rightly to on-the-job training. Then the tasks have immediate utility and are appropriately motivated. In school the connection is tenuous at best. I do not suggest here that mensuration and proportion should be eliminated from the school program. These two topics, central to so many applications of mathematics, belong in the curriculum; they belong there in general form, however, not in terms of each of their thousand and one specific applications.

Now let us finally turn to our thematic question: What are the basic mathematical skills? How we separate these basic skills from applications that are appropriate to on-the-job training will be considered in the next section. I address both questions by reference to the following diagram, in which the contents of the interior of the larger circle are my basic mathematical skills. Consider them in order of importance.



Mathematical Trust

This forms the central core of our diagram. It represents the part of mathematics that was often designated "structure" during the New Math era of the 1960's. But the word structure caused serious problems. It led to unfortunate stress on rules, definitions, formal logic. Here we mean something more relaxed: student belief in the consistency of mathematics, in the fact that today's ideas will last overnight, and in their ability to figure things out for themselves.

The tragic thing about mathematical trust is that it is all too often schooled out of students. Bob Davis has expressed this in the form of a pedagogical theorem: Young students always give correct answers. Obviously this statement has two unstated qualifications: (1) within the student's own conceptual framework, and (2) barring simple carelessness.

Consider in this regard two examples taken from interviews with young children by Eugene Nichols and Stanley Erlwanger and other members of the Project for Children's Mathematical Development at Florida State University:

1. Shown five counters separated by hand into groups of two and three counters, a first grader recorded what he saw as

$$5 - 23 = 0$$

2. Told that he could work from right to left, another first grader completed the exercise

$$\_\_\_ = 5 - 2$$

by entering -3 in the blank.

In the first case, the student was recording the two and three counters removed from the original five, and leaving none. His syncopated notation for two and three was 23.

In the second example, the student was responding literally to the instructions to work from right to left: Two minus five is indeed negative three.

Loss of Trust. Now what is the classroom response to these correct implications of the students' mathematical systems? Ninety-nine times out of a hundred it is negative: everything from raised eyebrows or a check mark to disciplinary action. The lessons that are sooner or later communicated to students are that mathematics is not consistent and that the name of the game is what the teacher wants, sensible or no.

This is a serious and central problem. Failure to appreciate the logic and structure -- merely other words for consistency -- of mathematics lies at the heart of most mathematical failure. Unfortunately, the standard school response to mathematical difficulty of any kind is to turn away from the problem entirely: "He doesn't understand, so we'll drill him on the facts." But what do the facts mean without understanding? They are long lists of three- and four-digit license numbers: 325, 4312, etc. And the very students who are least equipped are asked to burden themselves with these nonsense numbers. The attack on concepts and understanding is always harder, but it alone offers the possibility of long term gains.

Although this aspect of basic mathematical skills represents the central core, it is in no way separate from the ring of activities that surround it. In fact the interaction between the core and the next ring is two-way: The development of the other skills is supported by the structure, and the structure is established through the consistency noted in the other skills.

Using Calculators as Tools. One way, for example, that mathematical trust might be established (or reestablished) is by considering how calculators work, how they carry out basic algorithms. Today's electronic calculators have made many mechanical calculators obsolete and readily available. Retooling to allow four-digit totals on the cost indicator has placed thousands of gasoline pump mechanisms in this category. Bringing such devices into the classroom has helped students clarify such mathematical ideas as proportionality and to see mathematical consistency.

Calculators provide another quite different basis for pedagogical trust. They permit students to attack rather complex problems without bogging down with computations. This gives them a feeling of power and understanding many have never had before. And the understanding is not illusory, for they have had to determine the correct operation in order to utilize the calculator.

Computation

In many past summaries of basic skills, computation provides the full list. Today an argument must be mounted to include them at all. The ready availability of calculators -- including in particular the ubiquitous cash register, which plays a far larger role in modern society than has ever been credited to it -- must downgrade the importance of skills. Similarly, the slowly increasing use of metric measurement will -- if that clarion call is not really another false alarm -- reduce the need for computation with vulgar fractions.

Even granting these arguments, the first of which has been true for a decade or more, computation is still a viable mathematical activity. The justification is merely different today. I have already indicated how computation relates significantly to belief in mathematical consistency. It is also important to have students understand how and why calculations work in order to use calculating devices properly. Knowing how to calculate often tells you when to calculate by that particular process.

Still, calculating should be placed in proper perspective. The present rate of sale for hand-held calculators (now in multiple millions each year) suggests that computation should be taught in the form: "What if we were shipwrecked on a desert island and we wanted to...." Frankly I see nothing wrong in this. In fact I believe that it will work as well or better with students than the pragmatic approach that "Everything must be of immediate use."[3]

As calculation settles into its new and proper role in the next few years, I foresee stress on choosing among alternate algorithms rather than mastery in terms of speed and accuracy of only one. Students will still add, subtract, multiply, and divide, and will compute with common fractions (yes!) decimals and percents. But the mastery now must be of ideas rather than techniques.

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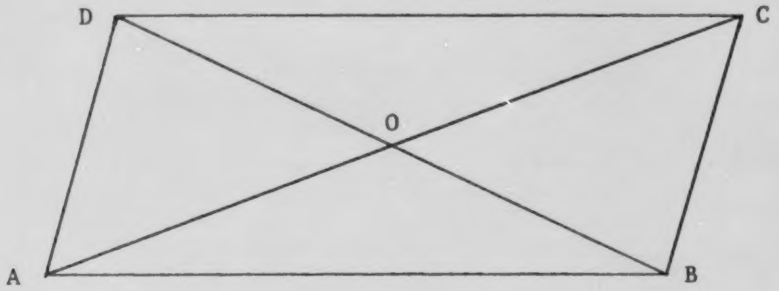
[3] Pragmatists today find themselves caught between the utilitarian skills they have always supported and the calculator. I suspect that they will have to abandon their old standby and turn to the new toy.

Geometry and Mensuration

Here again I see little change in the content as it relates to basic skills, but substantial change in the focus and the approach. In order to relate more intimately to the core of mathematical understanding, geometric ideas should be unified insofar as possible through transformations, and mensuration should be systematized through proportionality and dimensional analysis. The case has been made elsewhere for each of these ideas. I choose only to comment on each to suggest my thinking.

Transformations. If we define a parallelogram as a quadrilateral that is self-congruent (fits on itself) after a half-turn (about a central fixed point),[4] all the congruences of segments, angles, and triangles are immediately evident. In each case, the congruence is established, because one part fits on the other (the basic notion of congruence in the plane.) For example:



$$\angle DAB \cong \angle BCD$$

$$\overline{AB} \cong \overline{CD}$$

$$\overline{AO} \cong \overline{CO}$$

$$\Delta AOD \cong \Delta COB, \text{ etc.}^{5}$$

In each case we need only visualize the figure turned halfway around and fitting back on itself. Note how the simplicity and elegance of this approach provides the student with a further sense of mathematical faith.

Proportion. I have already noted in the nursing school example how proportion plays an important role in applications. It is difficult to find an elementary scientific problem that does not hinge on this notion in some significant way. Included here should be a systematic approach to ratio and proportion along the lines of that of Hartun and VanEngen, but without some of their formal overkill. Again, the elegance of the universal applicability lends strength to mathematical trust.

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[4] This corresponds to the definition of a parallelogram as a quadrilateral with a point of symmetry.

[5] I did not include the fact that corresponding lines are parallel under half-turns, because that would require an instructional regimen (or definition.)

Dimensional Analysis. The Wisconsin mathematician Creighton Buck tells of having discovered dimensional analysis for himself in the seventh grade. He made the mistake, he says, of communicating his discovery to an insensitive teacher whose response was that it wouldn't always work. Well, it does always work, and it provides a powerful tool to determine proper units and check procedures. One trivial example: Instead of memorizing complicated units like the number of cubic inches in a cubic foot (measures that we will be using for at least the rest of my lifetime), we need only cube the linear case. Thus:

$$12 \text{ in} = 1 \text{ ft, leads by cubing, } (12 \text{ in})^3 = (1 \text{ ft})^3\text{, to } 1728 \text{ in}^3 = 1 \text{ ft}^3.$$

The dimensions are cubed along with the numbers. (I have developed this topic more extensively in texts for grades 7 and 8. I refer you to them.)

Problem Solving

Applying computations appropriately is the real test of basic skills, and this is the section that appropriately closes our circle. Here I see the role of the calculator as pervasive. In the past students and their teachers have lost here what little faith they had in themselves. The teachers' litany goes, "I can teach them how to compute, but I cannot get them to solve problems."[6] In truth, the difficulty has been that students have lost the thread at two levels: reading and complexity. The calculator will reduce the complexity and allow a tighter instructional focus on procedure, interpretation, and reading.

The separation between problem solving and applications is a fine one, for the same exercise will often fit both roles. The difference is in neither degree nor type; rather it is in focus. Applications are pragmatic, focused outward toward specific tasks, specific jobs; problems, as we have defined them here, are focused inward on heuristics, general problem-solving techniques. Whenever possible, problems should be selected that do fulfill both roles, but there is no reason why nonsense or puzzle problems should be discounted when they serve here.

Robert Wirtz, one of the most important mathematics curriculum developers, has written a significant essay on drill and practice at the problem-solving level. In that essay he describes activities that respond not only to my concerns here, but also to a further unification of all four parts of my skills circle.

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[6] I have heard various forms of this litany in every grade from 1 to 13. In the latter resides the college calculus teacher who cannot teach max-min and related rate applications.

The ring is now complete, our four-region map stressing the interrelationships among the components across boundaries, and especially toward the core of understanding and trust. We must now turn to the serious problem of dealing with the more hazy and extensive outer space.

Real World Applications

Note at the outset that this annulus is excluded from basic skills. It is, in fact, *how* to "exclude" this region that poses perhaps the most serious problem of our basic skills definition. Failure to separate these applications, as I have suggested, drowns skills in the excess of including everything. In order to avoid this, basic skills should be identified as *those skills that provide a foundation for rapid acquisition of applications to specific areas*.

What we need, then, are some clear examples of training programs for specific applications. I suggest that development of such programs, quite possibly in comic book format, will help to define further and even specify the prior understandings needed by citizens to enter a variety of fields. Some of the applications programs might be: nurse, banking, office worker, cooking, construction worker, carpenter.

This proposal is clearly somewhat unsophisticated. Obviously a construction worker can be a day laborer or an architect, and the math applications vary over a wide range. But that should not deter us from picking a reasonable level of activity in a field from which to distill the mathematical activities. The next step will be to develop an instructional regimen to teach those skills on-the-job. The actual instruction of such a unit would then help to identify gaps in prerequisite (basic skill) knowledge.

Separation of applications from fundamentals may well prune the curriculum to reasonable essentials and take some of the onus off mathematics for the full range of society's ills. Then perhaps we can concentrate our attention on a more reasonable set of problems. We must learn not to try to include everything. We should select the applications to serve mathematics learning, including, of course, bringing the abstractions to life. In the terminology I have adopted, this means turning some applications into problems.

Summary of Comments and Recommendations

I set out here several observations about mathematics teaching and learning:

Public Conception. What the public (including and especially legislators) means by student achievement of basic mathematical skills is the demonstrated ability to calculate rapidly and accurately with the operations of addition, subtraction, multiplication, and division over the domain of nonnegative integers, with some limited extension to positive rationals: simple vulgar fractions, and particularly decimals as they apply to finance. This definition has, of course, obvious intellectual limitations. In particular, it is evident that these skills, so important to the parent whose sixth grade youngster cannot divide, do not stand alone. The adult with full command of these skills, but without knowledge of how and when to apply them, could not function mathematically.

Standardized Tests. Standardized tests have imposed all the limitations of this public view on the schools. The older tests especially limit the idea of mathematics to that of computation; even their so-called problem-solving sections are thinly disguised sets of computational exercises. Since these tests are so often used for comparison, their conservative force will continue well into the future.

The standardized tests encourage nonsolutions to contemporary problems of mathematics instruction, in particular, direct teaching for the specific test, a practice too common in our classrooms today. But the problem is deeper.

One of our major failures in thinking about education lies in internalization. The tests substitutes for and effectively blocks our thinking about what we should seek: Students who can take their learning out of the classroom and out of the school, and who will retain and develop this learning into adulthood.

Influence of "New Math." The contemporary political situation is that legislators everywhere wish to address problems related to computation -- as defined previously and as tested by the means suggested earlier to show "improvement" -- and are willing to back up their concern with dollars. Because of distrust generated by critics of the New Math, they are now unwilling to accept major modifications of this attitude. This reality must be considered in all of our deliberations.

Quality of Teaching. If we had strong, thoughtful, creative teachers in every classroom, this and other conferences addressing the "problems" of education would be few and far between. Fewer than one teacher in five fits this characterization, and many students pass through school without ever having been exposed to better than pedestrian teaching. If this personal observation is accurate, it implies the need for better teacher education. The current policies that change teacher education into an atomized regimen and change the teacher's classroom role into that of clerk will exacerbate, rather than respond to, this problem.

Quality of Materials. I suggest that curriculum redesign and the production of new materials predictably will have no effect on general achievement. Except in the hands of the designers themselves, and often not even then, these changes will only serve to modify some specific attainments -- replacing, rearranging, and so on. This is the effect so aptly described by Moise as "the shot heard 'round the immediate vicinity.' "

I believe that curriculum redesign is of great importance. If we wish to change the focus of mathematics instruction to respond to the computer-calculator revolution, for example, certainly we should reconstruct curriculum at all levels. But we must realize, even as we do this, that real intellectual gains will not be forthcoming, and that in fact a substantial number of classroom teachers will -- usually inadvertently -- compromise the curricular changes (for example, geometry in the elementary grades).

Loss of Computational Proficiency. The New Math of the 1960's does deserve some of the blame for lowered computational proficiency, but not nearly to the extent that critics contend. However, to suggest that this proficiency was never too high is to dodge the claims for improvement in this area that were the lingua franca of most of the curriculum developers as they prepared their materials.

There were two essential failures here. The first has been made much of by the critics: The failure to relate mathematics enough to the real world, to provide applications. The second and more important failure lay in the assumption that the classroom teachers, who after all had up to now focused all attention on computation, would provide that balancing view of the New Math abstractions. They did not.

What curriculum developers commonly fail to perceive is that pedestrian teachers teach any program with what has the outward appearance of a sort of vengeance (again in most cases with no volition involved), exaggerating every flaw and smoothing none of the rough spots. Thus, despite the obviously reduced amount of practice in New Math texts, few teachers saw any reason to supplement those exercises. I can provide much anecdotal evidence to demonstrate that both school systems and individual teachers who responded to this specific problem by incorporating even quite limited supplementary computational practice in their programs were rewarded with strong improvement in this area.

Aspects Overlooked by Critics. I hope that my previous experience with users of mathematics, particularly scientists, will not be repeated at this conference. If it is I will find myself again in direct opposition to them. I note the following blinders that have limited the thinking of many of the science critics of mathematics teaching:

1. Many of them would convert the mathematics classroom into an ancillary science classroom. This just doesn't work. The mathematics classroom cannot be reasonably assigned the role of meeting each day's specific science application. An extreme example may clarify my point. If we were to limit math to this role, we would have to provide 12 years of instruction in ratio and proportion (for school science), and then somehow during the summer vacation period just prior to college entrance, provide all the background for and the elementary concepts of the calculus (for college physics).

2. At the same time, most science teachers resent having to take any responsibility for teaching mathematics. (In this, they mirror the teacher of grade n who blames all problems on the teacher of grade n-1.)

What is needed here, of course, is cooperation. For a variety of reasons Paul Rosenbloom failed in his laudable effort to accomplish this in the MINNEMAST program. Still, we would do well to look at the great opportunity for positive interaction between the disciplines that the limited materials of that program demonstrate. The spirit of that effort suggests that much good mathematics can (and probably should) be developed in science settings, and that intimate contact between the two can contribute to intellectual growth, certainly the goal of both subjects.

Proposed Remedies

I turn now to my recommendations:

Recommendation 1: Teacher Education. In my postulate section I severely criticized contemporary classroom instruction, and the currently fashionable responses to this most serious problem. I wish to state clearly that I do not place the blame for this on the classroom teacher. This point is so important that I will spend some time on it. I suspect that too few of you have observed school instruction, and that fewer still will have seen teaching of subjects other than mathematics. I have, and I am profoundly shocked by what I have seen.

Consider the following scene that I claim is replicated thousands of times daily in classrooms across the country. The class is completing a reading lesson that has been carefully planned and organized. The teacher had developed vocabulary for the lesson. Those words remain on the chalkboard. The class had opened with an activity that stimulated interest and led directly into the reading for the day. For part of the hour the class divided into groups with every student actively participating. Now they finish up by discussing some ways that the story relates to their lives.

"Back to your seats," says the teacher. "Take out your math books. Turn to page 237 and look at the example there. See how that's done. Good. Now do the exercises on page 238. While you're doing them I'll work with a few at a time on difficulties left over from the reading lesson."

My point is this: The mathematics instruction is terrible, but the teacher could obviously teach mathematics as well as he taught reading if he knew how -- and knew that he should.

The case is scarcely more difficult to illustrate for secondary school, where the mathematics instruction is generally at least as bad, and college, where it may well be far worse. I am convinced that the teaching tradition has broken down at all levels in mathematics.

My first and highest priority recommendation to this conference, then, is that we seek ways that NIE can effectively address this problem of teacher preparation. Some more specific aspects are:

1. Developing preservice programs that will expose teachers to good models and encourage creative teaching. (Work at Ohio State University and at the State University of New York at Buffalo represent initial efforts in this direction.)

2. Developing inservice models for improvement of instruction. (I understand that Bob Karplus at Berkeley's Lawrence Hall of Science has done work in this area in science and is about to initiate a parallel effort related to secondary school mathematics.)

3. Developing a reward system for quality teachers. This could have monetary aspects (paying cooperating teachers for supervision of interns) but would more likely stress nonfinancial rewards. In England, teachers can work toward recognized national stature through a series of examinations. This system could be modified to provide a means to recognize quality teachers here.

4. Supervising the instructional staff, a serious problem in contemporary schools. Today's principal is a complete "writeoff." This situation should be addressed.

5. Establishing a new journal that would have as its central focus mathematical pedagogy. Despite their titles, such journals as Mathematics Teacher and Arithmetic Teacher scarcely touch on these problems.

Obviously, several of these approaches would individually drain NIE's resources if they were mounted on a broad front. I recommend the establishment of several demonstration centers to attack them locally, with adequate provision for wide dissemination of results.

These recommendations may seem unrelated to the specific thrust of this conference. I claim the contrary; that they are at the heart of the problem. If curricular revision or redevelopment is to be undertaken, a setting should be provided for a fair test of the materials. At this time there are no such settings available!

Recommendation 2: Calculators. Keep out of the standard activities. Leave them to the calculator companies. The National Science Foundation provides the perfect example of an agency wasting its money in this area by undertaking status and "needs" surveys and other "research" that will contribute nothing. Instead, NIE should support creative thrusts; the kind of work undertaken by such people as Henkin (and Kessner) at Berkeley and Kelly at Santa Barbara who use the calculator or computer in insightful ways. In particular, I suggest support for demonstration projects that would explore:

* Problem solving with a calculator by mathematically illiterate adults.

* Teaching basic computation by first introducing operations with calculators.

* Remediation by specific calculator intervention regimens.

The basic questions here are: What pedagogical contributions can calculators make? For how much can the calculator substitute? (A third question is equally important but I hope that NSF will sooner or later get around to exploring it: What kinds of curricular effects should computers and calculators impose?)

Recommendation 3: Science and Mathematics. As I indicated in the last of my comments, scientists and mathematicians should seek common ground. I hope that this effort will be reinitiated at this conference, and I recommend that NIE foster this cooperation at the curriculum development and school staff levels. In every case I know of, where school science and mathematics staffs have met together, the meetings have turned rapidly from initial sniping at each other to discussion of common and distinctive problems and, finally, to a search for means of solution.

The problems here must be faced seriously and realistically. There is certainly no call to define boundary lines between the two subjects -- and even more certainly no call to combine them! Instead, both mathematics and science would benefit from joint exploration of how to develop and reinforce such school concepts as measurement, ratio and proportion, and function.

Recommendation 4: Research and Development. I make this recommendation last for several reasons. First, so that others will consider it more thoroughly. And second, I believe that it is of less importance as a problem of *immediate* priority.

There are, I believe, two important questions already raised in this paper that need to be answered, and around which to initiate activity:

1. *Can concepts and skills be separated*? I am satisfied that the answer is negative, but I expect that some at this conference will argue otherwise. Even if we agree, I believe that there is need for exploration here. What mix is effective? How do we relate Piagetian concerns to this question? How do concrete objects apply here? (These are not new questions; I do suggest, however, that NIE should support creative study related to them.)

2. *Can applications be separated from skills and learning*? I do not mean a strict separation, but rather one of the type I described earlier. I cannot imagine mathematics developed without input from the real world. The problem is that unless the answer to my question is *yes* (in my limited sense), applications overwhelm mathematics: The physicist wants the calculus; the baker wants (thermometer) scale reading; the architect wants solid geometry; the notions store manager wants straightforward calculation; and so on.

I recommend here that my suggestion about the development of texts that summarize elementary applications to specific areas be pursued. By separating these applications in this way, attention can be given to the common underlying skills and concepts. Here is an area where cooperation with union and guild leaders in many areas would be most useful.

ACTIVITIES BASIC TO LEARNING MATHEMATICS: A PERSPECTIVE

Thomas A. Romberg

> All people do not agree in those things they would have a child learn--from the present mode of education all cannot determine with certainty to which men incline, whether to instruct a child in what will be useful to him in life, or what tends to virtue, or what is excellent; for all of these things have their separate defenders.
>
> Aristotle
> (Politics, Book VIII, Chapter 2)

As the title of my paper indicates, I have chosen to address a broader issue than just "basic skills and learning." I do this for three reasons:

First, I do not believe the participants of this conference can or probably should agree on exactly what we would have children learn.

Second, to me the phrase "skills and learning" is too restrictive. I believe an instructional emphasis is really what is needed.

Third, I believe there are activities related to the learning of mathematics we can agree upon, and we should use our collective influence to see that they are done.

Learning Mathematics

Underlying the important mathematical ideas are spatial-temporal facts about the world we live in. Unfortunately, what most people take to be mathematics -- the symbols, statements, propositions, and rules that appear in textbooks -- is only a written record of mathematica knowledge. Rarely do students have an opportunity to relate real world facts to mathematical ideas. Thus, my assumption is that students should develop processes, not simply rules as manipulation of symbols.

This paper is organized to (1) describe in some detail what I mean by "processes"; (2) discuss some of the instructional implications of having students learn processes; and (3) present what I think needs to be done.

Intellectual Processes

To solve any problem, a student carefully selects and uses "intellectual processes" that may bring about desired results. Descriptions of "intellectual process" usually show what happens when an individual uses concepts or skills to solve problems. Any suggestion that these processes are well understood would be an oversimplification. What we do know is that mathematical problem solving involves a complex set of interrelated processes. Some children learn the processes easily, almost as a natural development; other children experience considerable difficulty in learning one or more of the processes.

We also know that there are some processes children can use actively to solve problems in their world. When children enter school their reality is concrete and includes mainly the objects, people, and situations around them or those made real to them through pictures and stories. They have little in common with the abstract and symbolic world of the mathematics of adults.

The basic intellectual processes used in mathematics fall into five categories: relation, representation, transformation, validation, and structure processes. However, before describing these, I need to discuss "attributes." They exist only in connection with empirical objects such as physical bodies, electromagnetic waves, or persons. Usually, an object exhibits various attributes. A tone, for example, has loudness, pitch, and timbre. A child has height and weight, to name two. In examining a particular attribute, one neglects all the other attributes the object in question might have. For example, in examining the weight of an object, one neglects other properties such as shape and color.

Although many attributes could be examined in any mathematics program, there are some that are natural for use with children. There are attributes that can be measured such as length, numerousness, weight, area, capacity, time, and volume. There are also attributes that cannot easily be measured such as shape, color, location, and direction. All of the above are commonly included in mathematics programs.

Length is visually obvious, easily handled directly, and easily measured. Numerousness is usually the first attribute that is assigned a measurement. One can feel only gross distinctions among weights. A special instrument (balance) is needed to assist in making direct comparisons of weights. A student can visually distinguish between the sizes of two regions if they are grossly different in size or if they have about the same shape. Otherwise, the comparison of two regions on area is difficult. There is no convenient instrument like a balance or ruler to help. Capacity can be investigated by students when they try to find how much a container can hold. Volume, like area, can easily be distinguished if two objects are grossly different, but otherwise it is difficult.

In geometry, the attribute of shape plays an important role. The shapes of both three-dimensional objects and two-dimensional figures underlie many geometric concepts. Because color is the most perceptually obvious attribute, it is used to help describe and classify objects. Location of objects is investigated as the students learn to describe locations and to follow directions to locate objects. Movement and direction of movement are often used as a basis for developing integers.

Activities that ask students to focus on an attribute have problem-solving potential. These attributes are common to all students. Interesting problems about these attributes of objects and sets can be created. From such problems students can learn mathematical processes, concepts, and skills.

Relation Processes. There are nine basic processes used to relate objects on common attributes.

Describing is the process of characterizing an object, set, event, or representation in terms of its attributes.

Classifying is the process of sorting objects, sets, or presentations on the basis of one or more attributes into equivalence classes. Classifying is basic to mathematics, for it requires the student to look at how things are alike; if common attributes are identified, generalizations about the class can be made.

Comparing is the process of determining if two objects, sets, events, or their representations are the same or different on specified attributes. When comparing, the child focuses on an attribute to decide whether two things are the same or different on that attribute.

Ordering is the process of determining if one of two objects, sets, events or their representations is greater than (>), equal to (=), or less than (<) the other on a specified attribute. The process of ordering gives a background for developing the natural order of numbers.

Equalizing is the process of making two objects, sets, or representations the same on an attribute. There are two primary ways of equalizing: taking away from the larger, and adding on to the smaller.

Joining is the process of putting together two objects, sets, or representations with a common attribute to form a single object, set, or representation with that attribute. In the process of joining, one begins with at least two objects or sets and puts them together to make one object or set. This is most often represented with a sentence such as $5 + 7 = \square$ where the unknown is the sum. However, situations may be posed where one of the two objects or sets is unknown; these situations are represented by sentences such as $5 + \square = 12$ or $\square + 7 = 12$.

Separating is the process of taking apart an object, set, or representation that has a particular attribute in order to make two objects, sets, or representations, each with that attribute. Separating, as well as joining and equalizing, enables the children to solve problems that they will later solve symbolically with addition and subtraction.

The most common separating sentence is one of the form $12 - 7 = \square$, in which the whole and how much to be separated from it are known, but how much remains is unknown. However, in other situations one may also not know the whole, but know how much is taken away and how much remains ($\square - 6 = 13$) or know the whole and how much remains ($21 - \square = 7$).

Grouping is the process of arranging a set of objects into equal groups of a specified size, with the possibility of one additional group for any leftovers.

Partitioning is the process of arranging a set of objects into a specified number of equal groups, with the possibility of one additional group for any leftovers.

Grouping and partitioning are closely related processes. Both allow students to consider problems that will be solved by multiplication or division. In grouping, one knows how many in each group but does not know how many groups. In partitioning, students know the number of groups, so they deal out the objects one by one, giving each group the same amount and then counting the number in each group. When the action has been completed, in either a grouping or partitioning situation, it is impossible to tell which was done. In fact, both are represented symbolically the same way.

Both grouping and partitioning are used to convert from one unit to another within a system of measurement. For example, in changing 4 meters to centimeters, the student thinks of 4 groups of 100 centimeters, or 400 centimeters. Changing 20 quarts to gallons, the student thinks of the problem as how many groups of 4 does it take to make 20 or $\square(4) = 20$.

Grouping is also the basis of place value as the children group by 10s. Students could group a set of objects by 10s and represent it, for example, as 3(10) + 7. Because our place value system is based on 10, when grouping by 10s the notation is given a special name -- expanded notation. The student goes from this notation to the usual notation, or compact notation, 37.

Grouping is also used in connection with the addition and subtraction algorithms as the students regroup, for example, changing 3(10) + 7 to 2(10) + 17 to subtract in the problem:

$$\begin{array}{r} 37 \\ \underline{-\ 19} \end{array}$$

or regrouping after adding to change 7(10) + 14 to 8(10) + 4 in the problem:

$$\begin{array}{r} 45 \\ \underline{+\ 39} \end{array} \qquad \begin{array}{r} 4(10) + \ \ 5 \\ \underline{3(10) + \ \ 9} \\ 7(10) + 14 \\ 8(10) + \ \ 4 \end{array}$$

Partitioning is also often used to build an understanding of fractions. If a set can be partitioned without any leftovers, then each group is a fractional part of the whole set. For example, if 12 cookies are distributed equally to 3 people, each receives one-third of the cookies.

Representation Processes. The representation processes allow the student to progress from solving problems directly to solving abstractly. Students begin by solving problems directly with the objects or sets involved. In solving many problems, they gradually learn to use physical, then pictorial, and finally symbolic representations to help them. Representing not only includes going from the concrete to the abstract, but the reverse as well. For example, a student can represent the symbol 6 with 6 objects.

Physical representation is the process of characterizing an attribute of an object or a situation by using other physical objects. Consider the following problem: Suppose someone who cannot see the object wants to know how long it is. The student can cut a piece of string that is the same length and show it to the person.

Pictorial representation is the process of characterizing an attribute of an object by drawing a picture or graph. For the physical representation problem, the child can draw a picture of the length of the object or graph how many units long it is.

Symbolic representation is the process of characterizing an attribute of an object or a situation by using special symbols. Eventually the child can use units to measure the length of the object and then count the units.

To illustrate how the relation and representation proccesses are related, let us consider the process of ordering. To answer the question "Is your arm longer than your leg?," the student could find out by bending his arm and placing it alongside his leg (directly ordering). To answer the question "Is your waist as long as your arm?," the student may put a string around his waist and compare that length to his arm (using a physical representation). Or the student may choose to make one chain of links around his waist and another the length of his arm and graph the two chains to decide which is longer (using a pictorial representation). Finally, the student may measure around his waist in inches or centimeters, measure his arm in the same unit, and decide if both are the same length from the numbers alone (using a symbolic representation).

Although most of this discussion on representing revolves around arithmetic attributes, the child also learns to represent geometric attributes. For example, suppose a child has moved from one position to another. He may physically represent this movement by placing a rope between the two positions to show the path; he may draw a picture to show the movement; or he may use a more symbolic representation to describe the movement, such as three steps to the left.

Transformation Processes. Much of what is commonly considered mathematics involves learning the processes one uses to transform a mathematical sentence into an equivalent sentence. Three transformations, algorithmic, sentential, and structural, are used in the elementary grades.

An algorithmic transformation is a finite sequence of steps one uses to close an open mathematical sentence. The common algorithms children learn to add, multiply, and divide whole numbers are examples of such transformations.

A sentential transformation is a finite sequence of steps one uses to change an open mathematical sentence to an equivalent open sentence. For example: When 314 + ☐ = 843 is changed to 843 - 314 = ☐, a sentential transformation is followed. Such transformations are efficient in problem solving because they provide the student, as in the pictorial representation example, with a way to change an unworkable problem to one where an algorithm can be used.

A structural transformation is a finite sequence of steps one uses to change a mathematical phrase to an equivalent phrase. These transformations are the structural properties of a mathematical system. For example, the phrase 3(5 + 4) can be changed to 3 x 5 + 3 x 4 (the distributive property for multiplication over addition with whole numbers).

Validation Processes. In order to determine whether a proposed solution is the actual solution of a situation or sentence, one must validate. There are three basic ways to validate in mathematics: authority, empiricism, and deduction.

Authority validation is the process of determining validity by relying on some authority. For example, if a child checks the answer to a problem by comparing it with an answer book or with the teacher's answer the student is relying on authority.

When empirically validating, a student represents the sentence with objects, pictures, or other symbols to help determine its validity. For example, suppose for the problem 9 ☐ 6 a student puts > in the box. To determine whether or not 9 > 6 is a valid sentence, the student could represent 9 and 6 with cubes and visually show that 9 cubes are more than 6 cubes. Similarly, for the problem 6 + ☐ = 10, suppose a student puts 3 in the box. To determine whether or not 6 + 3 = 10, the student could represent 6 and 10 with pictures and clearly show that 3 is an invalid solution.

When students use an empirical representation to validate an open sentence that has been solved, they show whether they understand the sentence. In deciding whether or not a solution is actually a solution, students build up confidence in their ability to handle problems independently. Empirical validation is a powerful process that assists students in solving problems.

Logical deductive validation is the process of determining validity by a deductive argument that is based on agreed upon common notions, definitions, axioms, and rules of logic. This process is at the heart of advanced mathematics.

Structure Processes. Through these one examines and discovers the properties of a mathematical system. Discovering, for example, that a particular system has an identity element, or is commutative, is important in the study of mathematics. Such discoveries have provided mathematicians with powerful tools for unification.

Instructional Implications

It should be obvious that if the students are to learn processes, they should be involved in activities that are problem based. I am also convinced that the successful acquisition of mathematics is heavily dependent on acquiring specific concepts and skills, for they are needed in using any process efficiently. For example, children should learn common names for geometric figures and how to add three-digit numbers.

When it comes to teaching mathematics, concept and skill development has three characteristics that make it particularly appealing. First, some concepts and skills essential to success in mathematics can be identified and described in a reasonably objective manner. Second, the concepts and skills tend generally to be sequentially related. That is, certain ones must be taught and mastered before others can be introduced; this relationship has clear implications for diagnosis and instruction. Third, and perhaps most appealing, many concepts and skills are generalizable.[1] However, in teaching them, three points should be made.

1. Concepts and skills must be developed in order to solve problems. Again, learning the symbols, statements, and rules of mathematics should not be an end in itself. These should be learned as outcomes of solving problems.

2. More than arithmetic concepts and skills must be taught. This is not to deemphasize arithmetic. It is meant to emphasize concepts and skills from other branches of mathematics such as geometry, statistics, and probability.

3. Abstract concepts and skills must be based on meaningful experiences. I believe that students should be involved in the discovery or creation of mathematical ideas found in activities of their own, rather than simply assimilating the record of other people's activities. Brownell (1944, p. 37) expresses this point well:

> Learning is progressive in character. The abstractions of mathematics are not to be attained all at once, by some coordinated effort of mind and will. Instead, we must start with the child wherever he is, at the foot of the ladder or at some point higher up. Well chosen sensory aids reveal the nature of the final abstractions in a way which makes sense to the child. If he can work out the new relationships in a concrete way and can test their validity in an objective setting, he has faith and confidence from the start; and he is the readier to learn with understanding the more abstract representations of mathematics.

A child must be given opportunities to move slowly from the concrete to the abstract. Many of the usual basic skills, such as writing numbers, or basic addition and subtraction facts, are often taught before the underlying understandings have been developed. Research shows that "the greater the degree of understanding the less amount of practice necessary to promote and fix learning" (Brownell, 1956, p. 11). Drill and practice are important, but there should not be an abundance of premature drill. Short, quick drills, when a child is performing at a symbolic level, are important for increasing speed and accuracy but not for increasing understanding.

In summary, the subset of mathematics selected for schooling should be centered around carefully selected problem-solving activities. These activities should be designed to be real to students and within their intellectual capability. They should be focused on attributes and should require students to use one or more processes to complete the problem. Furthermore, they should provide meaning to the development of mathematical concepts and skills.

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[1] An outline of the concepts and skills that should be taught appears in Appendix A, available from the author at Wisconsin R & D Center, 1025 W. Johnson Street, Madison, Wisconsin 53706.

Needed Activities

Mathematics is translated into real world behaviors only when human beings tackle the tasks of using the variety of processes in problem situations. Mathematics is abstract and lifeless until a student is involved. Yet, while students make the mathematics come alive, they also add to instructional complexity because they bring to the activities the full range of their differences. One cannot assume that the same activities will be equally effective with all children, nor even that all children need to learn the same concepts and skills.

I believe the most important problem today is the inhumanity of many school systems. That some schools foster a nightmarish learning environment that is joyless and repressive has been well-documented.[2] Too often children are not viewed as human beings with individual personalities, interests, and desires.

Contributing to this situation is the inadequacy of mathematical materials most teachers use and how they have been conceptualized. Mathematics is not simply a collection of concepts and skills, nor is it encompassed by a detailed list of instructional objectives. One rationale for the modern mathematics is so that the "collection of tricks" to be mastered in the traditional programs would have meaning. Unfortunately, in too many instances the unifying notions introduced have now been reduced to "more tricks" to be learned that are more abstract and less relevant to students' reality than the "old tricks." In the process students now become less proficient at some of these "old tricks."

I believe what is needed to make mathematics come alive for more students can be summarized under seven headings:

1. Develop good instructional activities. Above all, we need more sets of good activities. The development of experimental teaching units (ETU's) written from a variety of consistent mathematical frameworks is critical. There are good activities in materials developed by USMES, CSMP, DMP, Nuffield, etc. However, more materials need to be developed that include activities directed toward the teaching of processes.

2. Support concentrated research on the learning of processes in the context of mathematical problems. As stated earlier, any suggestion that the learning of processes is well-understood is an oversimplification. Research on this topic needs to be continued and expanded.

3. Support concentrated research on mathematical learning as a result of instruction. More studies that attempt to capture the dynamics of instruction need to be carried out. To understand the impact of an instructional event (or a series of events), it is necessary to understand what happens during the event. Underlying the rationale for this suggestion is the dynamic nature of instruction. That is, many results and processes in classroom instruction are unplanned: The variety of participant responses often changes the content and

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[2] For examples, see Holt, 1964; Silberman, 1970; Soble, 1969; and particularly, on mathematics learning, Bereiter, 1971.

direction of the event; the "outcomes" are often embedded within the process itself; each event is occurring within a larger and complex learning milieu; and the apparent content of the event shifts and changes when viewed from different personal and time perspectives.

4. Support concentrated research on "teacher task structure." We commonly assert that the success or failure of any instructional program is dependent on the abilities and skills of the classroom teacher. Of the varied and quite diverse functions a classroom teacher undertakes during the course of the school day, instructional activities are of the utmost importance. However, as teachers translate the espoused curriculum of a school into a set of instructional activities, the teacher-produced "de facto curriculum" often is quite different from that intended.

One hopes many concepts and skills are being taught to school children, but, in fact, the students have never had an opportuntity to learn them. There are really two aspects to this problem. First, in attempting to meet the standards set by external agents, teachers often cover the content of a program by skipping large sections, thus leaving out concepts, explanations, and opportunity for practice. The tendency in mathematics classes to skip over the important ideas (such as explaining concepts) in order to get to the computational skills is too common. This is done in spite of the fact that if the concepts that underlie the skill had been well developed first, the skill itself would have taken relatively little time to teach.

Second, it is also apparent that for practical purposes, many concepts or skills, while covered, were not taught. In the Concept Attainment Abilities Project (Harris and Harris, 1973), mathematical nouns that teachers thought fifth-grade students knew were not understood by most students. Words such as numerator, denominator, dividend or quotient conveyed little meaning to most students (Romberg and Steitz, 1970).

Even if the concept or skill has been well covered in the instructional materials, it does not mean that the student has had an opportunity to learn it. What is being argued is that "The message is in the receiver and not the sender." If individuals do not receive the message the way the information was intended, they are likely to misunderstand the message. They have not had an opportunity to learn it as intended.

We simply do not knw very much about what tasks teachers actually perform, how those tasks are related, what influence their occurrence has on performance, and how the tasks can be changed.

5. Support concentrated research on teaching technology. Part of the problem of inadequate instruction stems from failing to capitalize on technology. Mathematics instruction too often implies quiet activity at one's desk doing a dull set of exercises from a workbook. Learning should not take place independent of others, particularly at the elementary grade levels. If one is to believe the developmental psychologists, it becomes apparent that to learn during the concrete operational stage, one must talk about the things with peers. This is one of the reasons I am critical of individualized learning programs in which the learning is assumed to take place independently of other human beings.

To this add the variety of materials now available and the need for such research is apparent. Then, if one considers the impact of the hand-held calculator, metric measurement, devices, television, or the computer, the need is critical.

6. Support innovative teacher-training programs. I am not a teacher trainer, but I find most of what is currently being written about teacher training, in general, and the training mathematics teachers, in particular, to be absurd. Surely, if some interesting information were found in 4 and 5 listed above, it should be translated into sensible training procedures. These procedures should be both for preservice and inservice training.

7. Support the development of research methodologies that can be used to investigate instructional programs. Research in education is, in general, too restricted in its conceptualization and methodology. Broadening the conceptualization of problems beyond the agricultural paradigm of treatments, controls, and yield is essential. For example, applying standard analytic techniques to studies involving assessment of the effects of one or more treatments involves three key assumptions: (1) treatment effects are additive, (2) treatment effects are constant, and (3) there is no interference among different experimental units.

As a consequence of these assumptions, the concerns of a research methodologist usually are to insure that (1) the experimental units differ in no systematic way (usually accomplished by random assignment); (2) there are fully designed treatments with well-identified intended outcomes; (3) there are objective assessment procedures related to those outcomes (assessment-content validity); (4) there are small errors of estimation (high test reliability); (5) appropriate inferential statistics have been used without making artificial assumptions; and (6) conclusions can be drawn with a wide range of validity.

But for dynamic instructional events, the assumptions for standard procedures are false. In fact, experimenters expect effects to be nonadditive and different across experimental units, and they desire interference among experimental units. The implications of this lead to differences in analytic methodology -- such as attempting to describe how experimental units differ; trying to describe the dynamics of treatment, how it grows, changes, and how outcomes (both intended and unintended) evolve; designing data-gathering procedures (objective and subjective) to illuminate the events; worrying more about validity than either reliability or generalizability; using "evolving" hypotheses; and relying primarily on suggestive descriptive statistics.

What Can NIE Do?

I would like to see NIE (and NSF) promote diversity and concentration. Conceptual diversity is imperative. Various well-argued and consistent positions related to each of the above issues should be supported. Now is not a time to demand an orthodoxy of thought. Finally, a concentration of intellectual and materials resources over a long period of time is needed if we are to truly have an impact on instruction..

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BASIC MATHEMATICAL SKILLS AND CONCEPTS

P. C. Rosenbloom

Some General Perspectives

An analysis of basic mathematical skills and concepts, with a view toward educational planning, must take into account the social setting. It would be different for Bali from what it would be for the United States.

A child entering school in 1975 may be expected to graduate from high school in 1987, from college in 1991, and reach retirement age after 2030. We cannot predict with much precision what sort of society he will be living and working in during this period, except that it will be changing even more rapidly than ours. During these 55 years, it is likely that most trades and professions will undergo several changes, and that the requirements for employability will also change. Most people will need to be retrained or reeducated in order to remain employable.

In contemplating the educational programs of the next 15 years, we must think not only about specific skills for which there is a foreseeable need, but also the tools and motivation for continued learning. During the last 20 years mathematics has entered significantly into the education of biologists, social scientists, linguistics, and business administrators. As automation proceeds, we may expect that the employment market for unskilled labor will virtually disappear, and that the educational prerequisites for the skilled trades will continue to increase. Paraprofessionals will be introduced in many fields occupying positions between the skilled workers and the professionals.

There will probably develop a diversity of educational patterns, both after eighth grade, and especially after high school, and a variety of techniques for individualizing instruction.

We must assume that the present depression and high level of unemployment is only temporary. In fact, the present discouraging conditions will probably steer many students away from demanding professions, and thus lead to manpower shortages when the economy returns to normal.

There is a well-known lag between innovation and implementation in education. It may take 5-10 years for a substantial improvement developed now before it is widely adopted. It may take longer to effect serious changes at the elementary level, and 12-16 years longer for the students to emerge into the world of work.

Therefore, it would be a dangerous mistake to focus too narrowly on present conditions in planning for educational research and development.

Finally, while the bulk of the work done in this field should aim at fairly modest and conservative improvements which could have a significant effect within a few years, a portion of the resouces should be allocated to more radical ideas.

Universal Needs

We begin with a list of skills and concepts which, we believe, every American of normal intelligence should know. While we make no attempt to rank them in importance, our annotations give some indication of priorities.

1. Translating from real world to mathematics and back. Without this all the rest is of no earthly use. This includes knowing what operations to perform on which data in order to solve a problem and how to interpret the answer.

2. Reading numbers, tables, formulas, and graphs. This is a minimum of literacy needed by citizens and workers.

3. Computation with rational numbers. This should emphasize approximation and estimation of answers and errors, more than complicated formal computation; it should also include experience with computing devices.

4. Measurement. Use of common instruments for measuring length and volume, time, mass, temperature, and some electrical quantities. This implies stimulation of schools to acquire such equipment. Application to mensuration and rate problems.

5. Descriptive statistics, simple probability. Computation and interpretation of means and measures of dispersion. Estimation of probabilities from frequencies. Elementary combinatorics.

6. Applications to jobs, citizenship, and daily life. Experiences showing the relevance of mathematics to a variety of situations.

7. Games of strategy. Should expose students to the ideas that: (1) given the rules, one must abide by them; and (2) the rules can be changed.

Needs of the Majority

I would estimate that about 60 to 70 percent of the population can and should learn the following skills and concepts.

1. Solution of algebraic equations. Linear equations with one and two unknowns. Quadratics. Approximate solution. Use of tables, graphs. Interpretation of answers.

2. Functions. Graphs, tables. Interpretation. Terminology.

3. Programing and use of computational devices. Logarithms, slide rules, calculators. Some experience with computers, their power and limitations.

4. Space. Properties of geometrical figures. Measurement and construction. Simple trigonometry.

5. Mathematical models and deductive systems. Real world → model → theory → prediction → test → revision of model. Experience with simpler models than Euclidean space.

6. Probability and statistics. Testing hypotheses. Decision theory. Significance. Binomial and normal distributions (intuitive idea).

7. Applications of mathematics. Exposure to a variety of applications to natural and social sciences, and vocations.

Supplementary Remarks

The analysis in the preceding two sections is based on what we know now about mathematics, teaching, and society. The mathematics background sketched here is an essential tool for further learning in virtually every trade or profession, which we predict will be necessary for almost everyone. Reassessment may be necessary every 10-15 years.

Most Americans must continue to learn beyond the age when the law keeps them in school. Moreover, the specific payoffs for this learning are not generally foreseeable during the school years. Consequently, the school program must be designed to make the students want to go on learning, for (regardless of the particular details and the degree of mastery) the program will be a failure unless it is interesting.

In summary, we have outlined what we know how to do and what we know must be done. It is a modest and conservative proposal since it is already being done to some extent. We are simply saying that it must be done well on a large scale.

Short and Long Range Needs

There are urgent needs which must be met now. Remedial programs for high school and college students, especially in urban areas, should have high priority. Many institutions around the country are struggling with the problem at great cost of time, money, and manpower. What is most desperately needed is software -- effective materials in a form suitable for individualized instruction.

Another high priority is that of improving the preservice mathematical education of elementary teachers. This improvement must be institutionalized so that it becomes permanent. While the designs and implementation of such improvements on a large scale are a short range need, they are a prerequisite for long range improvement of the educational system.

Needed Research and Development

If we wish to make progress, we must not only do well what we are already doing, but we must explore the possibility of doing new things.

Even though such an investment in research and development may not bring practical improvements on a large scale in less than 10-20 years, it is necessary in order to adjust to the rapid change of the surrounding society.

The work of the Cambridge Conference, Minnemast, CSMP, and others shows that it is feasible to teach perhaps 70 percent of the students of normal intelligence a working knowledge of the real number system by age 15. It is almost impossible to accomplish this on a large scale without the improvement of the education of elementary teachers mentioned above.

For the basic component of student teaching in preparing elementary teachers for major change, there must be a certain amount of curriculum experimentation in the elementary schools. A number of topics such as these would be worth exploring:

Vectors in grades 5-8. Informal introduction of vectorial concepts would make feasible a radical improvement in the high school gemoetry program.

Programing in grades 5-8. The execution of the programs could be simulated by hand computation or calculators.

Optimization problems. Experience with the use of mathematics to decide which is the best among various alternatives.

Science-mathematics modules. USMES already shows that this is feasible.

Mathematical games of strategy. Motivated drill in computation as by-product of playing games. Preparation for axiomatic thinking.

It would also be desirable to stimulate, by a program like NDEA, the development in schools of math-science labs. These should be equipped with calculators, as well as measuring instruments, mechanical drawing tools, and the like. Manuals should be produced on the use of the materials.

Finally, there is an urgent need for research on the mathematical needs of the skilled trades. This is prerequisite for the development of relevant curricular materials.

There should be a continuation of research on the psychology of mathematics. Besides the Piaget-type of clinical research, where the investigator tries to elicit the extent to which a child has attained a concept, I would like to see more done in tutorial research. That is, design some experiments on the teaching of mathematical skills to individuals and small groups. These should be directed at finding effective sequences of stimuli, variety of student responses, desirable branching, and the like. This research would be a valuable preliminary to classroom scale experimentation.

When I was directing the Minnemast project, I found that I could get a great deal of valuable creative work done inexpensively, by using bright undergraduate research assistants working with experienced teachers. I think that the inclusion of support for such students in research and development work would, with a modest investment, create a pool of personnel which would be a national asset.

RECOMMENDATIONS FOR THE EUCLID CONFERENCE ON BASIC MATHEMATICAL SKILLS AND LEARNING

Joseph H. Rubinstein

Introduction

Before attempting to address the questions asked of the conference participants, I must qualify my remarks with a brief description of my background. I am not a mathematician. I was trained as a natural scientist, and did research and teaching in biology before becoming interested in the education of elementary school children about 6 years ago.

For the past 3 years I have been developing an elementary school science curriculum for the Open Court Publishing Company, an assignment that has allowed me to work closely with the authors of the Open Court Mathematics Program, Carl Bereiter, Peter J. Hilton, and Stephen S. Willoughby. My duties include administering rather large and well-controlled fieldtesting and teacher-training programs in elementary school mathematics and science. I, therefore, speak as a biologist who has some experience with elementary school children and teachers, not as a mathematician or a mathematics educator.

What Are Basic Mathematical Skills and Learning?

Basic mathematical skills should encompass all of the skills that deal with numbers and the use of numbers that are necessary for functioning effectively in society.

Topics of a more advanced nature should also be included in the curriculum, in preparation for later work in college. I will omit the advanced topics from my list of basic skills because I think they are best left for the mathematicians to discuss. But I would like to make one remark about them. It should be possible to design the activities in which these topics are introduced in the elementary grades in a way that all children can enjoy and learn from them, although some children will understand more and be able to generalize the new ideas to problem-solving activities. Designing such activities and sequencing them with the other more traditional lessons will not always be easy or even possible, but it is a true challenge for the designers of curriculums.

My list of basic skills follows. For the purpose of this paper, I am interpreting "skills" as a kind of shorthand for abilities, understandings, knowledge, attitudes, etc., as described in a memorandum from NIE to conference participants.

Basic Skills

1. A firm understanding of the basic meaning and uses of numbers for counting, comparing, and ordering.

2. A mastery of the basic operations with whole numbers (addition, subtraction, multiplication, and division). Whatever other skills children learn, it is absolutely essential that they have the ability to calculate a precise answer easily and whenever necessary. In this basic skill I include not only the appropriate algorithms for calculating answers but mastery of the basic addition, subtraction, multiplication, and division facts.

3. The ability to use the appropriate operation (or operations) for solving problems. This skill includes the ability to recognize which problems can, in fact, be solved using mathematics and which cannot.

4. An understanding of when an approximate answer is appropriate for a particular problem, and the ability to make such an approximate calculation.

5. A firm understanding of magnitude with respect to numbers and measurements. This skill includes the ability to estimate the magnitude of measurements and know when such an estimation is appropriate as compared to an exact measurement. Also, it includes a firm understanding of the need for standard units of measure and the ability to make such measurements using the appropriate tools (rulers, balances, liquid volume measures, and thermometers). A complete familiarity with the metric (international) system of measures as well as the American system of measures is also necessary. Both systems should be learned independently, much as foreign languages are often learned, so that children, at least initially, are not asked to convert from one system to another, but are encouraged to think of each system independently.

6. The ability to organize or rearrange data intelligently. Included in this skill are the routine tasks of tabulating and graphing results, but at a higher level I include the ability to detect patterns and trends in poorly organized data, either before or after rearranging the figures, to show clearly such a pattern or trend.

7. Closely related to, but I think distinct from, the ability to organize data and detect trends (6, above), is the ability to extrapolate and interpolate and to know when such extrapolation or interpolation is justified.

8. The ability to use probabilistic ideas intelligently in ordinary applications. This basic skill includes an understanding of sampling techniques, the ability to describe a population in terms of simple statistics (mean, median, range) and, most importantly, the ability to act rationally when presented with the probability of a particular event's occurring.

9. The ability to recognize a particular answer as being absurd, contradictory, or otherwise illogical without having to calculate the correct answer. Stated more broadly, the ability to think intelligently using numbers and to solve problems imaginatively.

10. The ability to solve problems involving money, and the ability to use money in performing ordinary activities. I would include in this basic skill

an understanding of the relationship between income and expense, and an understanding of roughly how much money everyday items cost.

11. An understanding of the meaning of rational numbers, and of the relationship between fractions and decimals, including the ability to use them in all basic operations.

12. The ability to calculate proportions and percents, and the ability to use both of them intelligently in ordinary activities.

13. A firm understanding of, and the ability to use, such concepts as perimeter, area, volume, and congruence as applied to common geometric figures and consumer-type problems.

14. The ability to read and use a map.

15. An understanding of the meaning of signed numbers and the ability to add and subtract with them.

16. An attitude that mathematics can help solve real problems, rather than make problems solely for the sake of solving them.

What Are [illegible] Major Problems Related to Children's Acquisition of Basic Mathematical Skills and What Role Should the National Institute of Education Play in Addressing These Problems?

I will consider four major problems related to children's acquisition of basic mathematical skills. Before recommending what the National Institute of Education should do (if anything) to attack these problems, I will first list them.

1. Inadequate teacher education and training.

2. Inadequate parental education, training, and involvement.

3. Insufficient research about the specific techniques that have helped, or are likely to help, teachers teach specific topics in different types of classroom situations.

4. Inadequate curriculums. (This situation is changing and will change even more rapidly if the first three problems are solved.)

Following are my recommendations for approaching each of the problems. The recommendations assume that only limited funds will be available and that, therefore, the highest priorities should be given to those that are likely to have a short range impact on children's learning. With this in mind, I have tried to keep my recommendations practical. I recognize, however, the danger of shortchanging the basic research, which will be absolutely essential, if we are to proceed toward fundamental longer range educational reform.

Inadequate Teacher Education and Training

The overwhelming majority of teachers I have worked with are sincere, dedicated individuals who do a reasonably good job in spite of their preparation. The education and training that elementary school teachers and prospective teachers are getting can, however, be improved. There is a substantial need for preservice education in specific disciplines (mathematics, science, history, etc.). In fact, I hope that all teachers would be required to major in a specific discipline while at the same time studying educational philosophy and being trained in teaching techniques. Before a teacher can be effective, he or she must know the subject that is to be taught, and know it well. Obviously, elementary school teachers cannot be expected to study in detail all of the disciplines they are expected to teach. I do think, however, that intensive study in any one discipline will provide the ability to study other disciplines independently. There is also a need to start classroom training earlier than the junior or, as more frequently happens, the senior year of college.

Unfortunately, I cannot see what the National Institute of Education can do to solve these problems that is not already being done by colleges of education and other interested groups. There is, I think, general agreement that teacher education and training can be substantially improved. But I think a different type of approach for upgrading the standards of teaching, and one which is likely to bring more immediate results, should be pursued. I suggest that NIE begin a program to encourage the establishment of a system, or systems, of peer group accountability among teachers at all grade levels. For example, funds for a feasibility study could be made available to an organization such as the National Council of Teachers of Mathematics, and the results of this study could be widely disseminated and discussed. Models for establishing systems of peer group accountability already exist in other professions.

Inadequate Parental Education and Involvement

Most of us would agree that the home is more important than the school in determining the future success of children. Yet most parents have only a vague idea (if any idea at all) of what their children are learning in school and, more importantly, do not receive on a regular basis specific suggestions from the schools on how to help their children. Usually when such help is forthcoming from the schools, it comes because of some sort of crisis that requires immediate remedial work. How much better it would be if parents were enlisted routinely to help their children's studies at home.

In our own program we try to enlist parental support and involvement by supplying sample letters that teachers can adapt and send home to parents at different times of the year. In addition, we have developed a technique by which lending libraries for mathematics games are established in the classroom, and children are encouraged to sign out games to take home and play with their parents or siblings. I think that this is an area in which NIE could be involved.

I cannot recommend any single best approach, but offer the following two as being worthy of some consideration.

1. Provide funds to schools for the purchase of materials related to the mathematics curriculum that can be sent home to parents. When the practice of sending materials home becomes well established and has proved effective, the schools will probably continue to do this on their own and the funding can be stopped.

2. Provide funds to conduct mathematics workshops to be attended by parents, or by parents and teachers together.

Insufficient Research About Specific Teaching Techniques

There is a need for the development and dissemination of methods for teaching specific skills to specific children at specific times. In the past, too much research has been involved with trying to answer broad general questions about the way children learn and the way teachers teach, while neglecting more specific subjects. More research of this type will be of little use to classroom teachers and of only slightly more use to curriculum developers, although it will continue to be useful for longer range objectives. (I am certainly not opposed to basic research.) What is needed are the answers to more practical types of problems:

1. What are the methods that have proved effective for teaching place value to first graders?

2. What are some of the successful methods used by classroom teachers to get children to want to estimate answers by approximate calculation?

The answers to these and similar questions could be put to immediate practical use by classroom teachers and curriculum developers. Certainly, having such information available would not make teaching or curriculum development trivial; it would, however, help to prevent repeating the mistakes of the past.

Specifically, I recommend that a compilation of effective teaching techniques be prepared and widely disseminated. Beyond the routine tasks of assembling and organizing the information, the major problem will be to make the information highly selective. Methods will need to be found for certifying whether a particular technique is effective, and it is likely that such certification must come from data other than the type obtained from usual standardized testing.

Whatever the merits of standardized testing (and there are some), too much information will be lost if it is applied to this sort of task. Curriculum developers will want to know the chances of a particular technique's being effective for specific children, not for a school or even for most classroom populations. I also recommend that research toward finding better teaching techniques continue, but caution that, for the most part, this research should be aimed at very specific areas and should include techniques that are useful for both compensatory programs and for programs for talented children.

Inadequate Curriculums

I do not think that NIE should become directly involved with the development and wide-scale implementation of complete mathematics curriculums. This can be, and is being, done within the private sector. Moreover, better curriculums will be developed by the private sector when the three other problems discussed above are attacked and solutions found.

NIE should, however, support projects that promise to bring truly novel ideas into the curriculum. Such support should aim at producing curriculums, or parts of curriculums, that can be used as models by school systems when developing their own programs.

BASIC MATHEMATICAL SKILLS: SOME PHILOSOPHICAL AND PRACTICAL REMARKS

Israel Scheffler

Familiarity -- in life -- may breed contempt. In thought, it typically breeds complacency and misunderstanding. The very familiarity of educational concepts often masks critical features of the problems facing us, lulling us the while into a false sense of clarity and the fitness of things.

Thus, we divide the matter of education into familiar "subject" categories and think thereby to have simplified and clarified the tasks of teaching. The subjects are, after all, draw in directly from parent disciplines, each with its distinctive and authoritative purchase on the world, each with its characteristic methodology and set of truths. Every subject is intellectually homogeneous within, and separable from every other without. Subjects are for the knowing, and knowing them is a matter of mastering their respective stores of truth and acquiring the respective methodologies from which their truths have sprung.

Mastery of truths has to do with getting the appropriate beliefs; acquisition of methods and operations involves getting the right skills. For each subject, there are characteristic and peculiar truths as well as distinctive and appropriate skills. To find these and to state them is to produce a curriculum. What could be more familiar -- or more misguided?

Subjects are not, in fact, drawn directly or readily from their parent studies and parent studies are not all disciplines (Scheffler, 1974, Ch. 4). (Is social science a discipline? Is the study of English language and literature? Is history?) Neither adult studies nor school subjects are written in the sky. The former are arranged for the expedient advancement of investigations and researches, the latter for the facilitation of learning and teaching in particular contexts -- purposes that generate independent and powerful constraints.

Neither studies nor subjects are internally homogeneous nor are they wholly discrete from one another. Their aims, structures, methods, and boundaries change over time, and there are overlappings and branchings of various sorts at any given time. The "foreign relations" of subject areas at one time (generally of little interest to the specialist) are, moreover, of particular concern to educators, interested as they must be both in economizing educational effort and in broadening the student's intellectual and cultural perspectives (Scheffler, 1974, p. 89).

Nor is the concept of knowing -- taken as comprising acquisition of the distinctive beliefs of a subject and its distinctive methods -- anywhere near as complex and broad enough to capture the tasks of curriculum formation (Scheffler, 1965, pp. 106-107). The aims of education must encompass also the formation of habits of judgment and the development of character, the elevation of standards, the facilitation of understanding, the development of taste, and discrimination, the stimulation of curiosity and wonder, the fostering of style and a sense of beauty, the growth of a thirst for new ideas and visions of the yet unknown. The articulated truths and methods of a subject are raw materials; they have no fixed locus in the curriculum. They are given special forms of life by the curriculum design that puts them to use for educative purposes.

Finally, it is (to say the least) gratuitous to suppose that methods and skills are exactly correlated; the classification of methods is typically a "logical," epistemological, or normative matter, while the classification of skills is often a matter rather of the psychology of learning or cognition. There is no reason to suppose that methods or operations, as cataloged by disciplinary or subject specialists, are identifiable with skills, as conceived by psychologists of cognition. Nor is there any substance to the notion that there must be simple rules for translating methods or operations into the underlying psychological processes.

It is true that the word "skills" is ambiguous, often used interchangeably with "operations" or with "methods," or referring to what is classifiable by appeal to the latter, as, "capacities to perform such and such operations or to apply such and such methods." The skills of a subject, in the latter sense, are trivially derivable from a knowledge of its ingredient methods and operations. It is, however, fallacious to pass from the latter sense of the word to the sense in which "skills" refers to the basic processes of learning or cognition involved in applying a method or performing an operation. For educators, this fallacy is a significant hazard, for they must be concerned both with the intellectual methods of the subjects and with the psychological processes engaged in learning and applying them.

The Meaning of "Basic"

Let us direct some of the above reflections to the case of basic mathematical skills. The very phrase is difficult for a variety of reasons. I have suggested an ambiguity in the term "skills" as between "operations or methods," and "processes of learning or cognition." Consider now the force of the term "basic": Does the phrase refer to skills important (perhaps necessary) for mathematics, to skills peculiar to mathematics, or to both? These senses diverge. The ability to follow an argument, for example, is certainly necessary for mathematics, but surely it is not peculiar to mathematics.

Consider the specific ability to follow a mathematical argument: Is it not both important and peculiar to mathematics? The idea suggests, if it does not imply, that mathematics is internally homogeneous with respect to the argumentation it displays -- a point to be considered in the next section. Moreover, the idea begs the critical question as to whether the ability to follow a mathematical argument -- no matter how such argument be characterized -- is a separable skill from a psychological and educational point of view: Is it, indeed, optimally developed in islolation from nonmathematical materials, and is its sphere of exercise limited to the mathematical domain? If, in short, mathematical argument is peculiar to mathematics, it does not follow that the ability to learn, follow, or apply mathematical argument is peculiar to mathematics.

Recent neuropsychological studies have cast empirical doubt on the correlation of subjects with mental processes. Such studies have tended to destroy the a priori skill clustering fostered by traditional subject divisions: Certain symbol processing abilities appear to cut across these divisions in various ways, while other inherited rubrics appear to require new divisions (Gardner, 1974; Goodman, 1974; and Gardner, Goodman, and Perkins, 1974).

It is conceivable to me that very many (possibly all) of the psychologically significant skills important for mathematics may turn out also important outside mathematics, and that the particular perceptual, symbolic, inferential, mnemonic, questioning,[1] strategic, and imaginative capacities exercised in mathematics are also exercised outside it. Moreover, it seems overwhelmingly likely that successful performance in mathematics rests not only on general skills but also on general attitudes and traits such as perseverance, self-confidence, willingness to try out a hunch, appreciation for exactness, and others.

The Scope of "Mathematical"

Consider now the adjective "mathematical" in the phrase "basic mathematical skills." I have suggested that fundamental psychological processes connect mathematics with other fields of study. Here I argue that the subject itself is not all of a piece. Further, I conjecture that the oversimplified educational concept of a "subject" merges with the false public image of mathematics to form a quite misleading conception for the purposes of education: Since it is a subject, runs the myth, it must be homogeneous, and in what way homogeneous? Exact, mechanical, numerical, and precise -- yielding for every question a decisive and unique answer in accordance with an effective routine. It is no wonder that this conception isolates mathematics from other subjects, because what is described is not so much a form of thinking as a substitute for thinking. What it is, in point, is the process of calculation or computation, the deployment of a set routine with no room for ingenuity or flair, no place for guesswork or surprise, no chance for discovery -- no need for the human being, in fact.

Calculation is certainly indispensable to mathematics but it is not mathematics. When children are first taken beyond the sphere of elementary calculation to the stage of problem solving (perhaps in geometry), they naturally bring with them the impression that they are still learning the same subject. Yet here there are no routines for getting the right answer, here trial and error reign and there is ample scope for invention and surprise. The great gulf between mere calculation and problem solving occurs within the subject, not beyond it. The tasks set, the purposes envisaged, the rules and constraints of the game are of a fundamentally different quality and they are likely to evoke different applications and combinations of mental capacity. The notion of skill is, in general, not self-sufficient; it cannot eliminate needed reference to the nature of the tasks in question, their governing purposes, and expected styles of execution.

This division between computation and problem solving is internal to the school subject of mathematics. To elaborate it in teaching would help (I believe) not only to improve the process but also to break down the mechanical stereotype of mathematics, and relate the subject to other areas of creative thought. Other internal divisions are suggested by recent psychological studies, but are not as yet well understood. Certain sorts of injury to the brain may, for example, destroy the capacity to read while leaving intact the ability to recognize pictures, though both capacities are visual (Gardner, 1974). To what extent does success in geometry depend upon pictorial, as distinct from linguistic processing, capacities? In what ways indeed is a diagram like and unlike a picture, a map, an equation, or a word description? (See Goodman, 1968.)

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[1] Compare the emphasis on questions and the posing of problems by Stephen Brown and Marion Walter. When referring to problem solving in this paper, I mean to include also the phase of problem formulation stressed by Brown and Walter.

Both the division between computation and problem solving and the possible division between geometry and other branches of the subject are threatened by the popular conception of deduction. This conception deserves comment, for it offers the public a seemingly general way to override both distinctions and assimilate mathematics once again to a species of mechanism.

Consider an ordered list of statements comprising a deductive proof: Each statement is an axiom, or follows by single application of a stated rule from earlier statements. The chain is held together by necessity, the strongest conceptual glue: If the premises be true, the conclusion cannot fail of truth. Where is there any looseness or leeway? The whole is tightly made, an army of statements marching in order by command, a machine whose gears mesh inexorably according to fixed structural patterns. How can there be any mention of trial and error, of discovery and surprise, in the same breath with deduction?

The flaw in this example is that the proof is given at the outset. Though each of its statements "follows" by rule, the proof is not generated by rule. The determination of its character as a proof may indeed be made by mechanical routine. It does not follow that such routines exist for the construction of proofs. Indeed, it is demonstrable that no such routines are generally availalble. To find a proof is no merely mechanical matter, but an open and creative challenge, in which ingenuity and good fortune, trial and error, and, at best, heuristic maxims hold sway (Church, 1936; Rosser, 1936; Quine, 1950, pp. 190-191; and Church, 1942, p. 170 ff., esp. pp. 172 and 175).

Moreover, the use of any formalism requires intelligence in application. The problematic material to which it is applied must, ordinarily, undergo suitable preparation, a phase of "words into symbols" that is not itself governed by mechanical rules but rather by good sense and an intuitive grasp of the problem's context and description.[2] One can, in fact, impeccably run through a formal routine stupidly applied; the cure for stupidity is, furthermore, not formal.

Application of formalisms and problem-solving strategies always takes place in a context and for a purpose. Intelligent deployment in context requires not only proper management of the formalism but appropriate application; the latter involves accurate observation, sensible reading, logical analysis of problematic statements, translation into appropriate symbolic form, and eventual translation back to suitable statements. All such tasks belong to mathematical applications and also to virtually all other spheres of human thought.

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[2] See the section, "Words into Symbols," in Quine, 1950.

Comprehension and Skill

I have said much about skill, virtually nothing about comprehension. What sort of skill is that? Elsewhere I have argued that it is not a skill at all (Scheffler, 1965, pp. 17-21). To approach education as if it were always a matter of equipping the pupil with skills distorts our thinking. The category of skills has special features; these cannot be transported just anywhere.

A skill, for example, is capable of repeated exercise in separate episodes of performance, whereas comprehension is not thus exercised in performance. One who knows how to swim may swim every Thursday at 4 o'clock; can we say, comparably, that one who knows how to understand quantum theory understands it every Thursday at 4 o'clock? A skilled person may decide not to exercise his skill; a person who can play tennis may choose not to. A person with an understanding of quantum theory cannot, however, choose not to understand it. Nor can one speak of practice in the realm of comprehension as one does in reference to skills. One cannot develop an understanding of the quantum theory by understanding it over and over again, nor deepen one's understanding by faithfully repeated performances of understanding. One can tell pupils to practice writing out a proof; it makes no sense to tell them to practice understanding it.

If comprehension is not a skill, what is it then? Can understanding a proof, say, be merely a matter of checking its demonstrative character? Such a check would yield the conclusion that what purports to be a proof really is one. Would it, further, guarantee an understanding of the proof? If the answer is negative, the question is: What else could possibly be required as a condition of understanding?

The point is elusive, but I suggest it has to do with appreciating the generality of the reasons behind each step (Scheffler, 1965, p. 70 ff.). These reasons are, further, of roughly two parts. The first is deductive: Those that characterize a given line as axiomatic or else derivable, by a single application of a stated rule, from earlier lines. Comprehensive, here, requires an appreciation of the generality of the rule, an ability to recognize analogous cases and to apply it elsewhere. The second kind of reason is strategic: Those that characterize a given line as a promising step, in virtue of a certain strategic principle, toward the desired theorem. Comprehension, here, requires an appreciation of the general strategic principle, and it is also evinced in the treatment of parallel cases.

It is perhaps not misleading to describe deductive reasons as looking backward, while strategic reasons look forward. Strategic reasons enable us, not to judge the validity of the product, but rather the rationale of its step-by-step construction, thus to enter into the mind of the maker. They answer the questions that are often so puzzling to the student: "How did the author of the proof think to apply such-and-such rule to get the next line? Granted, the step is valid, but where in the world did he get the idea to take that step in the first place, and what made him think that it would bring him nearer the desired conclusion?" To such questions, it is no answer at all to be told that the step is valid. And without an appropriate answer, one can hardly be said to understand the proof in the full sense of the word.

Students who blame themselves for failure to understand, may never have been helped to see the special character of their questions, and the special nature of *strategic*, as distinct from *deductive* reasons. Poincaré (1952, pp. 129-130) speaks of the matter in terms of intuition and image, describing the needed insight as "seeing the end from afar." I have elsewhere described it as a matter akin to grasping the author's *motive* (Scheffler, 1965, p. 73), the embodiment of which is his strategy: What did he hope to achieve by this step and how is such achievement related to his final goal?

Understanding is not a skill but rather a state -- an attainment -- in which there are general ingredient capacities. It is a fundamental and important aim of education, because it places the particular item in a general framework of rules and principles. Understanding not only gives evidence of the pupils' right to be sure of their particular items, and hence, of their knowledge of them. It reaches out from these items to whole infinities of parallel cases, in which evaluations are to be made, but moreover, new efforts undertaken. Nor is understanding an all-or-nothing affair, because it may grow gradually with the attempt to see how to deal with new cases.

The very process of testing for understanding, furthermore, tends to develop it by forcing accommodation of the particular case with general principles (Scheffler, 1965, p. 73). The more reflective a grasp the pupil has of such principles and the more adequate they are to available cases, the less arbitrary the cases look and the more reasonable the principles seem. Also more adequate, moreover, is the pupils' orientation to new problems to be confronted.

Problems and Research

As suggested throughout the previous discussion, there is, I believe, much basic research to be undertaken, both of an analytical and of an empirical sort. There are also important studies to be conducted, of both a clinical and a practical kind. I shall comment briefly on these varieties.

As I hope to have shown earlier, the familiar categories in which educational thinking is cast are replete with difficulties. These difficulties are conceptual, but they critically affect the organization of practice. Analytical study of educational concepts needs to be undertaken, with particular reference to their mathematical applications.

The materials of mathematics need to be studied both in relation to the educational analyses just proposed, and independently. I hope the foregoing discussion has suggested the importance of the logical, or normative, analysis of mathematical operations and methods, as helping to set the aims of mathematical education. Such analysis should investigate the diversity of tasks and purposes embodied in the several areas of mathematics. The study of methods ought also to be brought into connection with fields other than mathematics to discover new relations of a logical and epistemological sort.

Neither form of analytical study just proposed is, of course, self-sufficient. Both need to be brought into communication with empirical considerations and inquiries of various kinds. I have mentioned, in particular, some recent neuropsychological investigations into cognitive processes, and the new articulations of skills and capacities to which they lead. Such studies, turned to the special concerns of mathematics, may well result in new and important pedagogical ideas.

The study of so called "disabilities" in reading and other language functions is one mode of access to an understanding of underlying processes. Analogously, the systematic study of "disabilities" and deficiencies in mathematical areas may reveal new insights into forms and limitations of comprehension, with pedagogical reverberations, and suggestions for improvement. I suggest a large series of studies of deficiency of all sorts in children and adults, including investigations into mathematical "trauma," illiteracy, and misunderstanding.

The relation of psychological studies of mental process to normative studies of mathematical method needs to be systematically investigated. The relation should, moreover, also be set in the context of general aims of education. For mathematics, as I have earlier argued, is not an island: Its linkages with all other areas of education need to be taken seriously, and studied systematically.

I have stated some conjectures about popular conceptions of mathematics and mathematical operations. All education is affected by prevailing attitudes and images concerning the content taught (Cronbach and Suppes, 1965, p. 125 ff.). Studies of public attitudes toward mathematics might reveal the sources of many difficulties, and perhaps point the way to some remedies.

Finally, I urge the study of teaching practice. What are the successful practices of good teachers? Why do they work? What skills do they embody? Such study ought to have a historical and comparative side and not restrict itself to local current custom.[3] But many good teachers now at work are no doubt doing good things capable of generalization. They are, however, unknown and generally cannot inform others of their work through publication. They should be sought out and studied. Furthermore, teachers should be encouraged to develop new practices, and educators to design new patterns of teaching. I am persuaded that the intuitive practice of teachers is an important -- perhaps the single most important -- source of new notions for the improvement of practice, and even for theoretical ideas.

The proposals for study put forth here require collaborative effort. Such collaboration is a difficult, but a crucially important, element in any program for advancing our knowledge and practice in education. Mathematicians, teachers, psychologists, philosophers, social scientists, educational theorists, and still others need to find appropriate channels for sharing ideas and learning from one another. To develop such channels would be a contribution of great importance.

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[3] For example, why did Poland produce such a dazzling array of logicians between the World Wars?

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PHILOSOPHICAL DEFINITIONS OF BASIC MATHEMATICS SKILLS AND LEARNING

Dorothy Strong

Mathematics education producers and consumers daily aline themselves with one of two philosophical definitions of basic mathematics skills and learning. Those subscribing to one of the two definitions may be described as either conceptual skills philosophists or computational skills philosophists.

Conceptual Skills Philosophists

Conceptual skills development philosophists define the role of mathematics education as teaching students to manipulate mathematical principles, ideas, and strategies. They minimize building computational skills.

Computational Skills Philosophists

Computational skills philosophists define the basic role of mathematics education as teaching students to use rules and rote memorization to manipulate symbols. Very little emphasis is placed upon meaning and understanding.

Conceptual/Computational Skills Philosophy

This paper proposes that the answers of conceptual skills philosophists and computational skills philosophists are immersed in the crescendo of an important emerging group: Conceptual/Computational Philosophists. If we are going to have a realistic answer that leads to effective programs, this answer will react rationally and objectively to the definitions of basic mathematics skills and learning given by the first two groups. Such an answer will combine the strengths of the two definitions to produce a collective conceptual/computational skills answer.

Both conceptual skills proponents and computational skills proponents identify the goal of mathematics education as developing mathematically competent students. Conceptual/computational skills philosophists recognize the limiting factors that minimize results produced when either conceptual skills or computational skills are developed at the expense of the other. They also see the multiplicative learning power that is generated when the two approaches are combined to form a conceptual/computational skills development strategy.

A conceptual/computational skills answer that points the direction for mathematics education will require the development of an unselfish working relationship between supporters of the conceptual skills philosophy and advocates of the computational skills philosophy. Combining the efforts of these important groups in mathematics education to generate a meaningful conceptual/computational skills design can produce students who:

1. Understand basic mathematics principles underlying mathematics computations.

2. Use computational skills effectively when dealing with basic mathematics concepts.

3. Are not hindered from learning basic mathematics concepts by computational skills deficiencies.

4. Are not hindered from gaining efficiency in mathematics computations by conceptual skills deficiencies.

The following graphs pictorially demonstrate the results that can be expected from programs embracing the three philosophies.

Figure 1 shows a limited results curve that occurs when computational skills are stressed while conceptual skills are neglected.

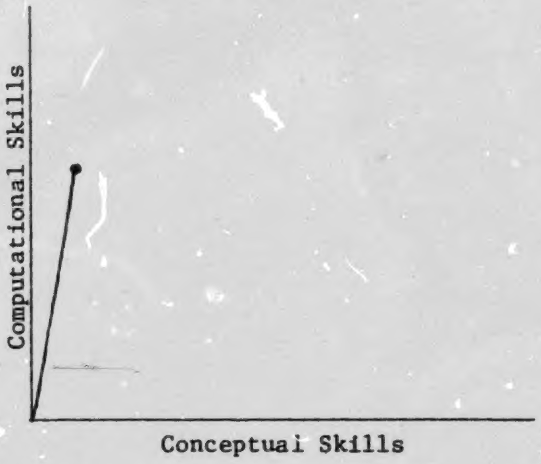


Figure 1. Results of computational skills based program

Figure 2 shows a limited results curve that occurs when conceptual skills are stressed while computational skills are neglected.

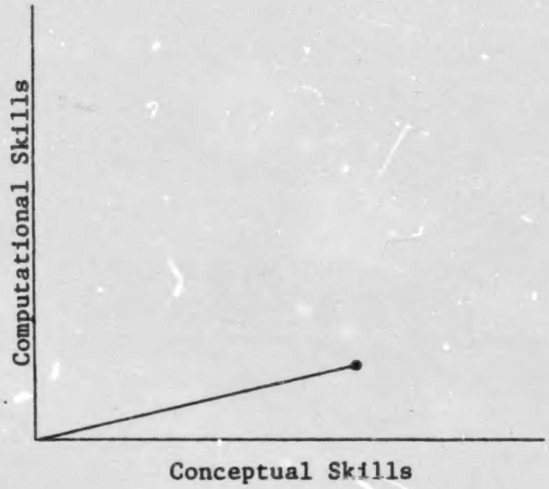


Figure 2. Results of conceptual skills based program

Figure 3 shows a continuous growth curve that occurs when both conceptual and computational skills are developed simultaneously.

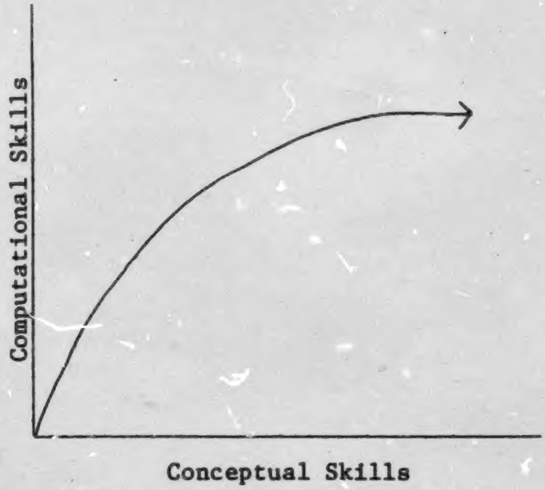


Figure 3. Results of conceptual/computational skills based program

A conceptual/computational skills-based program requires the following components:

1. A definition of basic mathematics principles that can contribute most to the development of students' understanding of mathematics as they contribute to efficiency in computational skills.

2. An effective method of combining conceptual and computational skills development.

3. A diversified teaching approach that uses the various teaching strategies designed for use with computational skills programs and conceptual skills programs.

Some mathematics curriculum development efforts of individuals and small groups have resulted in designing programs that embrace the conceptual/computational skills philosophy. One such example is the Chicago Prealgebra Development Centers that were designed as a solution to the serious underachieving problems of elementary school graduates. The solution that developed reflects a combination of curriculum, teaching strategy, and teacher training to concurrently produce dramatic improvement in students' computational skills and conceptual skills. Following is a summary of the components of the prealgebra program.

Five units of concentration were defined for the program:

1. Ratios and proportions.

2. Fractions.

3. Decimals.

4. Percents.

5. Measurements.

A major factor in the selection of the units was their importance as conceptual and computational foundations for algebra and other basic high school mathematics courses. Five basic mathematics principles serve as conceptual foundations for the total instructional program. These conceptual tools of the trade dominate the teaching methods used in the development of text materials in the program:

1. Place value.

2. One -- its names and properties.

3. Ratios and proportions.

4. Additive property of numbers.

5. Subtractive property of numbers.

A diversified teaching strategy provides the setting for the instructional program. It combines teaching strategies usually associated with either the conceptual skills philosophy or the computational skills philosophy to produce a conceptual/computational skills development teaching strategy. This diversified teaching strategy is known as the "Prealgebra LCD Technique." The LCD Technique combines:

L. Laboratory instruction ———————— Conceptual skills strategy

C. Classroom instruction ———————— Combined strategies

D. Diagnosis and remediation ———————— Computational skills strategies

Reading ———————— Combined strategies.

Combining conceptually/computationally-based instructional materials with conceptually/computionally-based teaching strategies produces tremendous student growth in mathematics achievement levels. These are reflected in performances in higher level mathematics courses as well as in mathematics-related courses.

What NIE Can Do

The advantages of a combined approach over unilateral attempts toward building mathematics competence demonstrates a need for concentrated efforts devoted toward defining curriculums embracing a conceptual/computational skills philosophy. To this end, it is recommended that NIE encourage the development of a Conceptual/Computational Skills Study Group. Such a group would include:

1. Proponents of the conceptual skills philosophy.

2. Proponents of the computational skills philosophy.

3. Proponents of the conceptual/computational skills philosophy.

4. Teachers experienced with teaching programs embracing the three philosophies.

The Conceptual/Computational Skills Study Group should be charged with the following responsibilities:

1. Identify strengths of conceptually-based curriculums.

2. Identify strengths of computationally-based curriculums.

3. Identify basic elements of programs asserting to be conceptually/computationally-based curriculum designs.

4. Use data gathered to define guidelines for a conceptual/computational-based curriculum.

A study group accepting the challenge to merge individual philosophies to develop a program design combining the positive advantages of each ideology will produce a new design capable of producing results that are equal to the sum of their powers:

Conceptual Skills Designs + Computational Skills Designs =

$$\text{(Conceptual Skills)}^{\text{(Computation Skills)}} + \text{(Computational Skills)}^{\text{(Concept Skills)}}$$

Our present embarrassing student achievement dilemma in mathematics education necessitates striving for these dimensions in student mathematics achievement.

A QUANTITATIVE APPROACH TO SCHOOL LEARNING

Clifford Swartz

Long ago in a land not so far away, the people decided that walking should be taught in school. It was clear that children should be taught to walk because the land was hilly and many of the paths were narrow and rocky. Instruction in walking was put into the hands of the dancing masters. Not only did they teach children how to walk, but the colleges did a brisk business in teaching people how to become dancing masters.

Dancing masters, by nature, did very little hiking, but they loved to teach the dance. At first there was instruction only in simple two-step, with more complicated waltzes and foxtrots for older children, and quadrilles at the graduate level. Later, during a period of curriculum reform, new courses were introduced in the theory of the dance, since many students were learning to walk by rote.

There was a widely adopted program in new-dance, as well as several attempts to teach unified dancing. However, many children came to hate dancing, and meanwhile, out on the paths, they stumbled along as badly as ever. In fact, some people complained that the children were not learning to walk as well as they formerly did. They did not dance well, either. There was some suspicion that the wrong people had been trained to teach the wrong thing.

Society has no need for all children to learn mathematics in the schools. There is, on the other hand, great need for all students to experience and learn the quantitative approach and techniques in many fields. Unfortunately, these methods are usually of secondary importance in today's mathematics courses. Also, the selection and training of people who prepare to be math teachers militates against professional experience, skill, and interest in quantitative applications. The profession has become obsessed with fads that purport to make math learning easier or more sophisticated. Most of these fads have no sound theoretical bases in educational theory or in mathematics, and they have cost the nation a great deal.

The Quantitative Approach versus Mathematics

Here are some examples that illustrate the distinction between a quantitative skill of societal interest and its analogue that is of concern in mathematics. For any given example, it is probably possible to point to a homework problem in a modern text, and say, "See, we have dealt with that application." My contention is that the main thrust of the math courses is not concerned with applications, although there may well be exceptions in certain topics or certain classes.

Another reaction to the given examples may be, "But those techniques were used long ago; nothing new is being proposed." Once again, I am criticizing most math courses that are being taught at the present time, not the many exceptions in the past or present.

Finally, it may be thought that my examples illustrate the difference between applied math and pure math. What I am trying to show, however, is not examples of applied math, but a quantitative approach to problems that arise in everyday life as well as in such academic fields as science and social studies. The quantitative approach is partly a frame of mind that has been developed from continual experience with measurements, numbers, units, and simple calculations. It consists not only of knowing how to apply math, but of knowing which math to apply, and to what extent.

The ABC's of math at the grade school level consist primarily of adding, subtracting, multiplying, and dividing real integers. There is no valid research showing that children in the grades learn to add and subtract faster or more accurately by being exposed to theoretical explanations of the processes. In particular, there is no indication that there is any benefit, practical or otherwise, in learning how to express numbers or do addition in terms of numbers to bases other than 10. There is no indication that skill in the basic processes is helped by learning how to describe groups of objects in the pseudo-technical language of sets, or in doing paper and pencil exercises using special school symbols for inequalities.

The type of adding and subtracting useful to grade school children is invariably an operation concerned with real quantities. The sum of a list of numbers should represent a total number of children, or the cost of a shopping trip. The difference between two numbers should be the amount of change to be received.

In a quantitative approach, grade school children would add and subtract numbers corresponding to objects that they themselves could count and measure. Such exercises would be part of most other school activities, not just the math period. The process of learning the sums and differences would combine tangible experience with rote learning and flash cards. Math texts and teachers have apparently been aiming at some deeper understanding, useless for the age level, and futile as preparation for future math study.

A quantitative approach to learning requires multiplication and division skills for purposes such as finding areas and calculating rates. The arithmetic processes should not be separated from the studies where those processes are required. Manipulation of the symbols should be preceded and accompanied by the study of the tangible problem for which the math is a model. Memorizing the multiplication and division tables should be done by rote, with only passing reference to the nature of the processes or the symmetries involved.

In contrast, current math texts try to teach students about the commutative nature of simple multiplication and division, in spite of the fact that the students have no occasion at this age to meet any math operation that does not commute. To explain the nature of multiplication, elementary math texts use various types of algebraic bracket symbols such as: 55 x 32 = (50 + 5) x (30 + 2).

Thus, a rote operation that could be mastered turns into a baffling additional concept that must be translated into operational terms, since the grade school child is too young to understand it.

A quantitative approach requires much experience in using appropriate approximations and obtaining answers with appropriate precision. This sort of experience is natural for the student to obtain at every level of learning if the method is not traduced in the classroom. All calculations should be preceded by rough estimates. At the appropriate age, students should obtain order-of-magnitude answers before starting more detailed calculations, and then consider how many more significant figures are useful. This attitude toward numbers corresponds to the emphasis on linking numbers and operations with tangible quantities and real processes.

In contrast, the effect of most math instruction is to insist upon detailed accuracy for all processes, without regard to whether fewer or more significant figures are needed. Another aspect of this attitude is that in the math classes answers can be (and sometimes must be) left in terms of π and $\sqrt{2}$. The implication of such rules is that the number has a reality of its own, and is not something representing a certain number of units that can be measured. There is a place for this practice and attitude, of course, but the primary emphasis should be the other way.

The major use of geometry in everyday life and even in technology is the determination of areas and volumes. It has been this way since the days of the Egyptians. In the real world there is more use of trigonometry as a circular function than as a set of relationships of the angles of a right triangle. The useful features of these subjects can and should be taught in terms of measurements and graphs of real objects and their functional changes.

The standard school courses in geometry and trigonometry emphasize the development of a logic system and the memorization of a network of derivation schemes. A touchstone of the consequences is the fact that very few high school graduates know, from their math classes, what happens to the volume of an object if you double the linear dimensions, or in what way the sine of a small angle is approximately equal to the angle.

At the grade school level, geometry has been turned into a silliness game involving a special language that insists that a plastic triangle is not a triangle, and that differentiates between lines and line segments.

Functional relationships and graphical representation are the foundation math for all technologies, and for all of the quantitative social sciences. There are only a few simple functions: linear, positive and negative power, exponential, log, sinusoidal, Gaussian. All of these are models of processes accessible to experimental analysis by high school students. Yet very few students get to use these functions in modern math courses. The drawing and interpretation of graphs is usually an isolated topic, with no attention ever paid to the graphical recording and handling of experimental error in the data points.

In grade schools, where children can easily learn to make and interpret graphs showing functional dependence and distributions, graphs are usually limited to bar charts. Even when graphs are studied, the data are given in the text and the whole operation is another paper and pencil abstraction.

The Theoretical Bases for Math Teaching

There is very little sound theoretical justification for most of the methods and topic sequences of math teaching. Most of the methods follow the cowpaths of tradition. Thus, we have fractions taught before decimals, and decimals taught before most students have matured enough to use them. In spite of claims about math labs, math is still almost universally taught as an abstract study isolated from the other subjects of the curriculum.

Some of the innovations in math teaching in the last decade are apparently based on someone's notions about how math is learned. There can be no other excuse for the introduction and emphasis on using number systems with arbitrary bases. Someone must have thought that by learning how to count in terms of base 3, students would come to understand the arithmetic processes in base 10. It is a senseless expectation, but teachers and students throughout the country were bullied into trying it. It is hard to say why the language of set theory was wished upon school children. The real business of set theory is at the college graduate level. The language is an affectation at any other level, providing no mathematical rigor or insight into its nature.

The soundest learning theory today, particularly in science and math for children, comes from the work of Jean Piaget. Anyone who has worked with children and listened to them must agree with Piaget's general conclusions about developmental levels. While many of the detailed aspects and implications of Piaget's theories are still subjects of research and debate, we know enough to plan curriculums that involve student abilities we are sure about. The major guideline is that students should always be faced with tangible experiences before the abstract models are introduced. In general, grade school students should never use math that is not an immediate model of something the students can see or measure.

The Selection and Training of Math Teachers

As long as mathematics is taught as an isolated subject, math teachers will receive training only in the rituals of the mathematics profession. Indeed, except for those who study in research universities, they will probably not even get to know the concerns and nature of modern mathematics. What they will not get in almost any institution is training in quantitative methods applied to any other field. A science requirement is normally satisfied by taking a 1-year course, sometimes in physics. In most places where there are two tracks in physics, the math majors take the weaker one. My own experience has been that there is good reason for this; the math majors are frequently weak in the math analysis required for the upper level physics course.

If mathematics in the schools turned into the use of quantitative analysis in many different fields, then math teachers should become knowledgeable about those fields. They should be familiar with measurement and analysis in the sciences, including the statistical analysis used in the social sciences. Such a requirement could be affected by extending the minor course requirements, or by designing courses for math majors that involve real laboratory experience.

Some of the Quantitative Skills That Should Be Taught

Quantitative analysis should be done, at all levels, as a series of approximations. In senior high school, this should be done in terms of order of magnitude and subsequent significant figures. Numbers should be given in terms of power-of-10 notation. The significance of significant figures and sensible error treatment should be stressed in terms of real student experience.

Since much of the math manipulation training should be done in connection with science or social studies, numbers and variables should come from real data, should be expressed in terms of units, and should be hedged by error limits.

The decimal system should be linked to the use of the metric system. In this way it is possible and advantageous to use and manipulate decimals before the formal treatment of combination of fractions.

All measurements and relationships should be graphed, starting in grade school. The graphs should portray and emphasize functional relationships, including distributions.

Algebraic relationships and derivations should be accompanied by their graphical representations, and be models for tangible processes that can be immediately dealt with by the students.

The geometry of familiar objects should be analyzed, and the areas, angles, and volumes measured and calculated. Geometrical theorems and formulas should be demonstrated with measurement and plausibility arguments, not formal proofs.

The sinusoidal function should be treated like the other simple functions, as a model of tangible processes, to be modeled in turn by graphs. The application of the sinusoidal relationships to geometry should be treated in terms of experiments with actual objects, and by the use of plausibility arguments instead of formal proofs.

The study of probability and statistics should be tied to course work in other subjects. Real data should then be analyzed primarily in terms of graphs. There should be more stress on this subject than there is in the present curriculum.

The Formal Math Courses That Should Remain

So far we have been talking not about math courses, but about training in quantitative techniques that all students should receive. There should also be formal math courses, particularly in high school, for the same reason that there should be music courses. It is part of our cultural tradition, and is particularly interesting for some students. Certain types of math preparation are useful for those students going on with math studies in college, although the preparation seems to be useful mostly in developing the esoteric quality known as mathematical sophistication.

At the high school level, math topics should be chosen that are used now in business, science, or early college work. Math teachers should not try to anticipate future math uses; let the colleges do that. Do not assume that high school math must prepare students for the concerns of modern math in college or research. There is time for such studies in college, at an age when the students have the experience to tackle them.

For advanced students in high school, it is reasonable to study calculus only to learn the simpler operational techniques of differentiation and integration. These should be linked to graphical models of the operations. Proofs should be limited to plausibility arguments. Much of the emphasis should be in terms of the analysis of physical processes, such as velocity and the work done by a variable force.

A course involving a sequence of logical proofs, such as the traditional Euclid's geometry, is great fun in its own right for many 15-year olds. Some of the more recent versions of this course have increased the use of pseudo-rigorous language and steps, taking some of the fun out of the study for many students without really increasing the validity of the rigor. There should also be more treatment in the eleventh or twelfth grade of analytical geometry, including the use of cylindrical and spherical coordinates. In all of these geometry treatments, there should be a minimum of formal mathematical derivation and a continual attempt to show how the math is a model of real situations.

Real mathematics, of course, is much more than a model for physical or social relationships. The small percentage of students who will study advanced math in college will learn the nature of math in good time and when they are old enough to appreciate it. We really do not need to worry about this select group anyway. They will take care of themselves, starting at an early age. It may well be, of course, that an enthusiastic and well-prepared math teacher might want to offer optional courses in pure mathematics for the students who have the time and inclination. Such offerings should not be confused, however, with the standard school math course, and, of course, they have nothing to do with the training in quantitative techniques that should be built into broader courses for all students.

What Would Be Needed To Introduce the Quantitative Approach

Some of the new science courses at grade school and junior high level have exercises in quantitative analysis. Most of the commercial math texts include examples of applications, although these frequently use artificial and unrealistic data and require only pencil and paper work. Social studies texts make very little use of quantitative analysis, even of graphs.

A project to introduce quantitative skills into the schools would first face the task of producing guidelines of expectations for various grade levels. These expectations would be defined best in terms of specific examples, chosen from present texts or created for the occasion. At any rate, part of this task should be to survey present texts and actual practice to see what is available. Almost certainly a development of suitable materials in many subject areas and at every grade level would follow. In some cases these materials might be provided as supplements for existing courses. In other cases, mini-courses in science or social studies might be produced.

One way or another, there ought to exist a set of standards for minimum proficiency of quantitative skills. There ought also to exist a sufficient amount of quantitative curriculum material so that a teacher, a school, or course designers could build it into existing or future courses.

EUCLID CONFERENCE PAPER

Ross Taylor

What Are Basic Mathematical Skills and Learning?

There could be many different answers depending on the context, to the question posed above. For example, basic mathematical skills needed by a nuclear physicist are quite different from those needed by a concert pianist. Basic mathematical skills for enlightened citizens are different from the minimum mathematical skills necessary for survival.

For purposes of this discussion, I shall focus on minimum essential mathematical skills necessary for a citizen to function in society. This set of skills would need to be greatly expanded for many career needs, and for a person to be considered mathematically enlightened. The mastery of skills mentioned here could be considered as potential minimum graduation requirements. I believe that minimum essential skill objectives should be defined as narrowly as possible to permit the greatest amount of freedom by students and teachers to explore those topics in mathematics that they find interesting and challenging. We must be sure that, in the process of defining minimum essential skills, the minimum program does not become the total program.

First, I do not believe that any person or group of persons should generate a list of objectives to be used as graduation requirements for all schools in the country. Rather, the specific objectives should be selected locally. However, the input of persons with expertise in mathematics, in learning theory, and in curriculum development and implementation, is highly desirable in the development of a process for selecting objectives. Such a process should allow for factors operating to influence change in terms of which objectives are considered essential. For example, factors currently influencing change include the forthcoming conversion to the metric system; the current impact of the wide availability of electronic calculators; the impact of national, state, and local assessment; and growing information about which concepts cause the greatest difficulty in learning.

In 1972, a committee of teachers in Minneapolis developed a set of preliminary guidelines for behavioral objectives in basic mathematical knowledge minimum essentials. I shall outline the types of objectives included in that list. I present this list because it provides at least an indirect input to the Euclid Conference for classroom teachers who have done considerable thinking about minimum essential mathematical knowledge. Furthermore, although the skills and concepts listed are quite elementary, many students graduate from high school without mastering a good share of them. With the current "back to basics" thrust and the emphasis on accountability, the failure of many students to master minimum essential skills is a matter of increasing concern in the schools.

Computational Skills

Whole Numbers

1. Addition up to five addends of up to four places.

2. Subtraction up to four-digit numbers.

3. Multiplication up to three-place multipliers.

4. Division up to two-place divisors.

With the availability of inexpensive calculators, perhaps even less calculation could be required.

Fractions

1. Concepts emphasizing the relation of fractions to parts of a whole or to subsets of discrete sets of objects.

2. Addition and subtraction of fractions with the same denominator, with denominators restricted to the numbers 2, 3, 4, 5, 8, 10, 16.

3. Addition and subtraction of fractions with different denominators, with denominators restricted to the numbers 2, 4, 8, 16. (In everyday applications, the major application in adding and subtracting fractions relates to measurement. With the conversion to the metric system the need for this skill may no longer be necessary.)

4. Multiplication of fractions and mixed numbers, with denominators restricted to the numbers 2, 3, 4, 5, 10. (The times are indeed rare when the everyday citizen needs to multiply fractions involving other denominators. We might even simplify this objective by assuming that a calculator is handy and that the fraction can be converted to decimal form before multiplying.)

5. Division of a fraction or mixed number, with denominators restricted to 2, 3, 4, 5, or 10, by a whole number restricted to 2, 3, 4, 5, 10.

Decimals

1. Converting between decimals and common fractions with denominators of 10 or 100.

2. Adding and subtracting amounts of money less than $100.

3. Multiplying an amount of money less than $100 by a whole number less than 100.

4. Dividing an amount of money less than $100 by a one-digit whole number less than 100.

5. Rounding to the nearest dollar or the nearest cent.

Percent

1. Relating a percent to a shaded portion of a 10 x 10 grid.

2. Making simple conversions between percents and decimals and between percents and fractions with denominators of 2, 4, 5, 10.

3. Solving problems of the type "x percent of y is what number?"

   (The other two forms of percent problems are not included because they are much more difficult for students and have much more limited application.)

4. Making simple approximations involving percents.

Measuring Units

1. Measuring lengths, using an arbitrary unit of measure.

2. Identifying simple units of customary measure and of metric measure (e.g., yard, pound, foot, degree, quart, mile, inch, meter, liter, gram).

3. Making simple equivalent conversions within the customary system and within the metric system (e.g., 1 foot = ____ inches; 1 meter = _____ centimeters).

4. Using a ruler to make measurements to the nearest eighth of an inch or to the nearest centimeter.

5. Performing the four operations with "denominate numbers" involving measures of time, liquid, volume, and weight.

6. Reading a Fahrenheit or metric thermometer.

7. Giving the area of a figure composed of a union of unit squares.

Reading Graphs and Tables

1. Determining from a bar graph or a line graph the magnitude of any entry on the graph.

2. Determining information from a table.

Applied Problems

1. Solving simple ratio and proportion problems.

2. Solving simple time, rate, distance problems.

3. Finding the perimeter of a rectangle or a triangle, given its dimensions.

4. Stating the area of a rectangle, given its length and width.

5. Substituting given values into a formula and evaluating. (The formulas should be kept simple such as $P = 2\ell + 2w$ or $I = PRT$.)

6. Finding the average of less than six numbers.

7. Estimating measures of angles and comparing sizes of angles.

8. Approximating solutions to simple problems.

9. Stating correct amount of change to be received after a purchase.

10. Solving applied computational problems involving one operation.

11. Solving simple applied computational problems involving two or three operations.

Of course, no list of objectives can describe everything we would like to have our students accomplish. As a teacher expressed to me recently, "What we really want is students to be able to think." In effect, we want them to be able to face new situations, and at least be able to attack the problems they will encounter in a wide variety of real life situations.

Our friends in vocational education tell us that each of our students leaving schools today can expect to have from three to seven different types of jobs in their lifetimes, with the higher number more likely. Consequently, rather than emphasizing preparation for specified occupations, we should emphasize preparation for change. This means that not only should our students learn mathematics, they should also learn how to learn mathematics. Furthermore, they should experience success in learning mathematics so that they will realize that they have the capability for more learning.

Finally, as stated in the philosophy of the mathematics department of the Minneapolis Public Schools,

> An objective of equal importance to that of learning necessary basic concepts and skills is that of developing a lasting interest in mathematics. Learning mathematics should be exciting and challenging. If we can make mathematics interesting for our students, they will continue to study it and will have the opportunity to remedy deficiencies at a later date.

What Are the Major Problems Related to Children's Acquisition of Basic Mathematical Skills and Learning?

Following are four areas of major problems related to children's acquisition of basic mathematical skills and learning:

1. Lack of motivation by students.

2. Inadequate teacher preparation.

3. Inadequate or inappropriate resources.

4. Lack of clear definition of what is to be accomplished and a systematic procedure for accomplishing it.

I shall deal briefly with each of these.

Lack of Motivation by Students

In many cases, lack of motivation results from lack of support in the home, as well as an inability of students to identify with that which is being taught in mathematics classes. Perhaps two of the greatest problems we have are to recognize that many of our students do not have strong support at home, and to find ways of either building that support or compensating for the lack of it. If students are not motivated to learn what we are presenting to them, we must reexamine our offerings to determine what does motivate the students. We should manage to creatively bring students in contact with learning experiences that are interesting to them and at times when they are ready to learn.

Inadequate Teacher Preparation

Particularly at the intermediate grade level, many teachers lack the necessary mathematics background to teach mathematics to their students successfully. Furthermore, the fear and dislike of mathematics by teachers with weak mathematics backgrounds are, unfortunately, often transferred to their students. On the other hand, especially at the secondary level, there can be a lack of knowledge about how mathematics is learned, particularly by students who may have learning difficulties, weak mathematical backgrounds, or a negative attitude toward learning mathematics. In light of these conditions, certification requirements for teachers of mathematics need to be reexamined and in many cases changed.

Today, declining school enrollment and other factors have caused a decrease in the number of teachers entering the profession. Consequently, in order to affect what is happening in the classroom, a major effort must be made in the area of inservice education of teachers. While inservice opportunities can be provided by the schools or by colleges and universities, some of the most effective inservice programs result from cooperative efforts.

Inadequate or Inappropriate Resources

In some cases, there are insufficient quantities of human resources in the form of teachers and teacher aides, or of physical resources in terms of classroom space and learning materials. In other cases, sufficient funding to support these resources is available, but the funds are not used in a way that will maximize learning. If you ask teachers what is most needed to help their students learn, they will most likely tell you "smaller class size." Because salaries are by far the largest part of any school budget, the matter boils down to one of obtaining sufficient funding and then giving prime consideration to basic skills in the allocation of resources.

Persons attempting to influence the allocation of resources within school systems should realize that decisions are often made more on the basis of the pressure that can be brought to bear than upon the stated goals of the school system. Teachers realize this, so they organize and bargain collectively for salaries and working conditions. Various parent groups apply pressure for allocations ranging from new athletic facilities to special education programs. At the local level there appears to be a need for organized pressure to obtain and allocate resources for learning the basics in the academic disciplines.

Teacher effectiveness can be improved through utilization of teacher aides to give individual help to students and to relieve the teachers of some of their clerical chores. With greater emphasis on activity-oriented mathematics, more mathematics learning space is needed. Today, in many cases, this problem can be solved through creative use of classroom space that becomes available as a result of declining enrollment. On the other hand, recent sharp increases in costs of learning materials and supplies are causing severe budgetary problems.

Within the past few years, ESEA Title I funds have begun to be used for instruction in mathematics as well as for reading. In many cases persons who administer Title I or other specially funded mathematics programs do not have a background in mathematics education. This can result in some rather questionable decisions with respect to mathematics programs. I feel that guidelines for federally-funded programs in mathematics should be revised to insure tnat persons with backgrounds in mathematics education have sufficient input into the decisionmaking process.

Lack of Clear Definition of What Is To Be Accomplisned and a Systematic Procedure for Accomplishing It

In many cases we hear statements of goals for improving basic language and mathematical skills that are merely platitudes because they do not relate to specific objectives or to means for determining whether or not those objectives have been attained. At each level, starting with the individual student, to the individual classroom, the school, the school system, the State and the Nation, we need a clear definition of what we are trying to accomplish and a specific means to determine whether or not we have accomplished it.

In effect, we need a systematic approach to the learning of basic skills. If that approach is employed, we should be able to obtain meaningful evaluation results to determine the effectiveness of various programs, procedures, and materials. Norm-referenced tests often used to evaluate programs are not suitable for a systematic approach to learning because they do not measure student progress toward specific objectives. Some form of criterion-referenced testing is more appropriate. A systematic approach does not imply the teaching of isolated skills and concepts one at a time through some sort of individualized program. A more integrated program may be more effective.

What Role Should NIE Play in Addressing These Problems?

In my opinion, NIE should play the same role on the national level with respect to these problems that the mathematics supervisor plays at the local level. The supervisor's job is one of stimulating the development, implementation, and evaluation of promising curriculum practices. The practices that evaluation show to be unsuccessful are either revised or discarded; those shown to be successful are given broad dissemination. Exchange of information is facilitated to avoid duplication of efforts in developing new programs, and to stimulate the implementation of those practices showing the greatest promise. NIE can offer the greatest service to the nation by helping local districts and states clearly specify their objectives, provide appropriate means of evaluating their programs, and disseminate evaluation results for various practices and programs.

A priority for NIE should be the provision of information that will help the decisionmakers at the various levels obtain optimal utilization of existing resources. There is an urgent need for NIE to obtain information that will influence the way Title I and other federal funds are spent. A broad definition of basic skills beyond the bare minimum essentials can be useful in stimulating more creative expenditures of Title I funds for mathematics programs.

If NIE is to continue curriculum development efforts, it should consider funding projects based in the schools. I am firmly convinced that the schools contain a wealth of untapped talent, that projects based in the schools are as close as you can get to reality, and that such projects have a headstart when it comes to implementation. In addition, school-based research projects should be given consideration. Questions faced by decisionmakers in the schools should be addressed as research questions. The results of such research could contribute significantly to improved instructional programs.

SOME THOUGHTS ABOUT BASIC MATHEMATICAL SKILLS AND LEARNING

James W. Wilson

This paper could have been labeled "random thoughts" or "incomplete thoughts" because my thinking about the problems under consideration at this conference is certainly in transition. In my professional role I have seen the problems from various perspectives. As a taxpayer and parent, I am dismayed that "what is" and "what could be" are so far apart, but admit that I do not have very good information on either.

If there is a theme to my paper, it would be a call for considering many options. We must have diverse, alternative programs. We must consider various approaches and perspectives in investigating our problems. For example, the answer to the question of what are basic skills needs the perspective of mathematicians, educators, psychologists, methodologists, supervisors, laymen, teachers, etc. Some of these groups are represented at the conference; some are not and some may be misrepresented.

Likewise, our proposed solutions to problems identified in answers to the second question of the conference, in my view, must reflect exploring alternatives. There must be diversity of effort, sharing responsibilities across levels of the educational enterprise (schools, universities, research and development agencies, government) as well as the pursuit of alternative solutions.

What Are Basic Mathematical Skills and Learning?

There needs to be a frame of reference for this question. Unfortunately, the statements from the NIE, intended to stimulate our thinking on the question, create confusion. For it seems that NIE wishes this conference to address the total mathematics program of the elementary school (i.e., "abilities, understandings, knowledge, and so on"). Somehow, the label of "basic skills" implies to me some restricted domain of necessary or minimal educational goals. I will leave to others to elaborate on the intricacies of appropriate definitions. I will assume the conference is addressing the mathematical competencies desired for each individual to function effectively in the highly technical society of today and the near future.

A thorough answer to the question would lay out a set of assumptions about what is mathematics and why we teach mathematics. This philosophical position would provide the rationale and criteria for selecting basic mathematical skills and learning. A list of basic mathematical skills and learning would then follow. One is almost led by the question to go directly to a list, but lists are difficult to evaluate without knowing the rationale and criteria.

I believe several thrusts are at work in mathematics education today, most of them with a considerable history. For instance, one thrust is built around a belief that children will search out and develop essential mathematics skills in the course of working on significant real world problems. Therefore, schools should provide the context and environment for encountering real world problems. The Unified Science and Mathematics for Elementary Schools (USMES) program is one proponent of this view. Historically, the core curriculum movement and other embodiments of incidental curriculum were similar. The degree to which such programs have failed may be more a function of the specific instances and uses than the philosophy.

Another thrust, perhaps similar to the incidental curriculum or problem-oriented curriculum, concentrates on social utility. The criteria for selecting basic mathematical skills and learning would concentrate on what mathematics does the child use and/or what will he or she be likely to use.

Another thrust is based on criteria inherent in the subject matter -- the structure and interrelationships within mathematics. Historically, Brownell advocated meaningful learning; the new mathematics movement drew upon the structure of the discipline. The advocacy of developing computational algorithms based on understanding is within the same framework.

Finally, the rote drill and practice of basic facts, operations, and algorithms has its advocates from the drill theory curriculum of the 1920's, right through to the predominant mode of operation in many classrooms today.

Unfortunately, we judge many things in today's society by the pathologies of some part of the system. Many are quick to condemn all of the Federal government because of Watergate; "bureaucrat" is somehow a derogatory word in many people's vocabulary. Any approach to identifying basic mathematics skills and learning could have its pathologies, and I place much of the criticism of modern mathematics programs in this vein. Inordinate attention to nondecimal number bases, modular arithmetic, or the language of sets, pedantic use of language and terminology, or rote learning of terms are all counter to the intent and purpose of experimental programs of the 1960's.

What Are the Major Problems Related to Children's Acquisition of Basic Mathematical Skills and Learning and What Role Should NIE Play in Addressing These Problems?

I want to start my answer here with a note of extreme pessimism. I do not believe this conference, or the NIE programs that may follow from it, or the various projects reported on here, will have any really noticeable short term effects -- either positive or negative. The most critical problem is in the schools, broader than any attention to essential skills.

Our schools are in chaos. Social promotion has made a mockery of standards. Too much has been shifted from the home and the community to school (or teacher responsibility). Teachers are so occupied with service and paperwork activities that they do not have time to teach. A recent column by James Kilpatrick ("Equal Time for Teachers") documents some of the response he got from teachers when he criticized the "educational establishment" for declining test scores. Hell, yes, test scores are declining -- because the schools have become situations where teaching is impossible, standards are nonexistent, and discipline is missing.

Need for Better Research

Research on how children learn basic mathematics skills and concepts is needed. Obviously, quality research in terms of significant ideas and problems is what is needed. We need to investigate all the relevant factors of a problem area systematically, through sequences of studies, replications, and research teams. Identifying a significant problem is the first priority; then, the appropriate methodologies can be selected. (I fear that NIE cannot live with such a restriction -- the methodologists will most likely be in control.)

There is also a need to explore different research methodologies. For example, the interactive clinical interview and the teaching experiment methodologies illustrated in the Soviet Studies of the Psychology of Teaching and Learning Mathematics ought to have application or adaptation to research problems. Granted, many journals probably would not publish articles based on such studies because it is the wrong methodology (for the journals). On the other hand, the first step is to produce some good quality studies.

We need a broader view of research. Perhaps the folklore linking "research" with particular methodologies argues for some other terms. As an administrator with responsibility for approving research workload, I believe a broad range of disciplined inquiry activities should be included. This includes exploratory investigations, materials development, evaluation, literature review, interpretation, criticism, preparing position papers (if I had research workload to assign to myself, preparations of a paper on basic mathematical skills and learning would be a valid research effort), status studies, or experimental studies. A concerted attack on the problems of basic mathematics skills will include all of these activities.

Some mechanism is needed to describe what is really happening in mathematics classroo[illegible]t materials are being used? How are they being used? This conference cannot do it. With due apologies to a couple of supervisors, there is not a real elementary school teacher participating. What is needed is some systematic study of classroom practices. There are school systems, for instance, that have done extensive work with coordinated mathematics and science curriculums (for example, San Antonio). What is the result? How extensive is local "curriculum development?" How widespread is the use of open classrooms?

Need to Study Testing

Perhaps one area (problem) where some potential leverage for change exists is in the area of testing. The various standardized tests have a tremendous effect on school decisions (Research question: What effect?). The tests are political weapons as well as essential instruments of school policy making. Yet, we have had no parallel to the curriculum experimentation of the past 20 years in mathematics achievement testing. Studies are needed to test development practices, uses of test data, alternatives to current testing program, and assessment procedures. Support of experimental test development, in the same sense that we support experimental curriculum development, ought to be encouraged.

Interpretive studies of assessment data or achievement test data should be done. I have a bias on this point, but I cite the work of the NCTM Project for Interpretive Reports on National Assessment as an example. (Summaries of the 1972-73 NAEP Assessment and some interpretations, which are the only general overviews of the NAEP mathematics assessment, were published in the October 1975 issues of the Arithmetic Teacher and the Mathematics Teacher. NAEP's own reports have been limited to (1) computation, and (2) consumer mathematics.) However, the presentation of results of studies, evaluations, or assessment is not sufficient. Those of us who are concerned with mathematics education must be concerned with critiques and interpretations. Policy implications cannot be left to chance, or left exclusively to persons outside of mathematics education.

Examining the Use of Calculators

I expect considerable discussion at this conference on the impact of the handheld calculator. Publishers are rushing into print with materials; school systems are experimenting with uses and policies, trying to cope with the existence of devices. We need systematic, well-conceived studies of various uses of handheld calculators. If the handheld calculator can revolutionize the structure and organization of the curriculum -- as many proclaim -- then groups should be at work, exploring the various alternatives. Frankly, I am yet to be convinced of what the Federal role here should be. Whatever, we must examine diverse alternatives.

The private sector -- publishers, manufacturers -- are perfectly capable of profitably meeting some needs. Local school systems have the responsibility and talent to meet other needs. There are plenty of opportunities for experimentation and research by those having such responsibilities in the schools, universities, and professional organizations.

Perhaps our attention to the handheld calculator is too limited. What are the longer term implications for handheld devices as they become more sophisticated? It is almost certain that in the next few years we will move to the point that handheld devices are the input-output medium for computers with extensive computational and information retrieval capabilities. What are the educational implications? Obviously there are many. My point here is that we should have some groups working on future-oriented studies.

Conclusion

We sometimes behave in mathematics education as though some ultimate solutions to our problems are to be brought about by new curriculums, or revised teacher training, or some new technology sensibly used, or . . . . We make promises to parents, politicians, or pupils that a panacea is at hand *if* only the cooperation, the money, or the commitment can be provided. My view is that whatever progress is made, we will have to convince both our clients and our employers that incremental changes may be significant. Moreover, we will *not* produce final answers to the questions on basic mathematical skills. Rather, we should adopt a point of view of continuing assessment of the problem on philosophical, educational, and empirical grounds.

We must recognize that the resources of NIE are finite. But our problems in exploring basic mathematical skills and learning are not just a Federal or an NIE responsibility. There is plenty for all of us to do. Let's get on with it.

WHERE DO WE GO IN MATHEMATICS EDUCATION?

Robert W. Wirtz

A Bit of Fantasy

Most elementary school children do not see any good reason for wanting to do mathematics.

While only a few have the inclination or courage to verbalize their doubts about the system, they must all wonder why they should do that stuff in the textbooks. And is there any intellectually honest answer for the 7-year old who asks the forbidden question: "If I don't enjoy it, why do you insist that I study mathematics?"

Answers about future usefulness do not satisfy normal youngsters. Pointing out the need to make change, to avoid ripoffs at the candy counter, only prompts the answer "I know that already"...or "OK, I'll learn that, but what about the rest of the stuff?"

Many years ago, my second-grade teacher had an honest answer for me -- "If you don't, I'll flunk you." But today all children know about "social promotions."

"Well, you don't want to be stupid in arithmetic, do you?" may well bring the rejoinder "My father claims he can't even balance his check book -- and my father's not stupid."

Now it just might happen -- at least in the world of my fancy -- that this 7-year old might take the offensive and launch into a lengthy soliloquy.

"Hey, do you think I didn't learn anything before I came to school?"... (You're not expected to answer.) I learned to talk, and the only help I got from grownups like you was that they let me listen to them. In the beginning it was a jumble -- a vertible cacophony of sounds"...(This is a precocious second grader.) "And let me tell you, I never would have made any sense of it, but when I wanted orange juice, I didn't want milk or water or crackers or to have my diapers changed. I got the notion that making special sounds could be more productive than crying and pouting.

"So I decided to make sense out of the noises. And as I was making a lot of progress, about age 2 or 3, the trouble started. I asked my mother to read me the story she 'readed' me yesterday. She told me I should have said 'read' -- and it sounded like the name of a color. Of course now I realize I had invented a new word that fit the patterns I was hearing. Maybe I should have filed for a patent -- but, I suppose there is prior art to show I wasn't the first.

"Anyway, I learned to crawl, to walk; I toilet trained myself and I learned to tie my shoestrings with a bow, among other things. I learned to read adult 'body language' -- I could tell when to run and when I was welcome. I learned a lot of neat ways to get what I wanted and to get out of what I didn't want. I learned 17 different ways to get the attention of anyone around me; some of them were pretty neat -- they still work.

"Now let me tell you something else; adults weren't very helpful. The most they did was let me listen to them, watch them and touch them -- and they listened back to me, watched me and touched me.

"Now I'm going to let you in on a secret -- 'cause I think you're too grown up to remember. I was able to perform these prodigious feats of learning with a very simple but effective strategy: I first decided what I wanted to do: then I embarked on a determined course of trial and error. When I decided I wanted to get around without waiting for adults who weren't always there or took me where I didn't want to go. I began thrashing around until -- wow! -- I was somewhere else.

"It was always the same -- decide on what you want to do, and plunge into trial and error.

"And then I got sent to school. It was plain from day one that adults have little or no respect for the fact that we kids have already learned lots in our own way. The 'school message' came through loud and clear: (1) someone else is always telling you what your goals ought to be, and (2) 'trial and error' is rejected as a childish way to find out what you don't know -- in school it's 'listen and remember.'

"Now I'll make you a bargain. I'll learn more mathematics than your stupid tests can keep up with if you just change your behavior in two ways: (1) give me a hand in this goal-setting business, and (2) after I'm convinced something is worth doing, please let me 'figger it out' as they say on the block...like I 'figgered out' that language mess once upon a time. One other thing, if I'm working toward some goal you've given your blessing to, and all of a sudden I think I see greener pastures up another valley, please don't fence me in."

A Political Position

There is a large population who feel that Standardized Achievement Tests measure development of the "skills and learning" they expect schools to foster in children. It is unquestionably a substantial majority of parents, schoolboards and other taxpayers. In the end, they will control.

In spite of this fact, "new math" set other goals and assured this large population that SAT scores would, after a while, creep upward. When this did not happen, an elder statesman of the "revolution" tried to reassure an angry committee of the state legislature that the "new math" children had a better foundation in mathematics even though SAT scores had reached new lows. The committee was not impressed.

Currently a well-funded effort has gone on beyond the goals of society and has set out to persuade children to push out along the cutting edge of qualitative analysis. Plans are to harness the power of "the tube" and to bypass schools, taking mathematics directly to young consumers. But, again, this is not what most people think of as "school mathematics."

(Note: Let us be thankful there are pioneers who see far beyond the mundane and meagre minimum standards set by prosaic schoolboards -- adventurers who are willing and able to open new frontiers. And let us be thankful that in the infinite wisdom of the great and good Fathers and Mothers in Washington, provisions have been made so that frontiersmanship is economically feasible. We can be less anxious because these pioneers are well fed -- though it remains to be seen whether affluence and pioneering are compatible.)

Enter another and more pragmatic group. They do not challenge society's clearly expressed goals; they deplore the inept, ineffectual unscientific institutionalized way education has pursued these goals. They offer versions of PPBS adapted for education and, guided by behavioral psychologists, they develop "management systems" that, applied conscientiously, must produce the desired SAT results. This offer is so attractive that school administrators have welcomed it as manna from Heaven...and administrator's Nirvana!

If the pudding from such a recipe had been edible, we would not be here today. Others would be in our places exploring ways to improve on successful "management systems."

But this is more of a wake -- a bit premature perhaps -- for a scheme that failed miserably to produce what society wants. Perhaps they failed because proponents of "management systems" did not realize that children are not Ford Motor Co. employees, nor soldiers -- that they are not Pavlov's dogs or Skinner's pigeons. Society is holding the managers accountable in terms of SAT scores and not in terms of "products" generated. The collapse of "performance contracting" heralded the fate that lay ahead.

Consider another strategy -- another game plan. Since society wants improved SAT scores, let us accept that as one of the measures of success. This does not mean, however, that we must be trapped into the behaviorists' fatal scheme of "teaching for the tests."

Most critics of SATs can agree that computational skills in mathematics ought to be subordinated to problem-solving skills and attitudes. But it ought to be possible for us to put together a program of interesting and relevant mathematical porblems for children to consider -- problems that would lead to development of computational skills and knowledge acquisition.

Already two Title III ESEA projects (both in Berkeley, California) that adopted this approach showed SAT results of more than 1.5 months gain per month in school. (These were K-6 projects involving 2,000 children whose "expectancy" was 1.0 month gain per month.) And the overpopulated "bottom quartiles" thinned dramatically as an increasing number of students moved over the Great Divide. The bimodal curve grew much less well-defined.

This can be shrugged off as another round in the numbers game. Agreed!... but what a political coup!

The standard textbook was shelved; the traditional scope and sequence was scrapped; teachers were sensitive to the affective needs of learners. While some teachers and parents had expressed initial concern for such innovations, the SAT results allayed all their fears -- skepticism gave way to enthusiastic support.

It's a high risk tactic that requires of humanists a solid faith in their convictions. It can provide an effective alternative to nonproductive "management systems."

A Basic Strategy

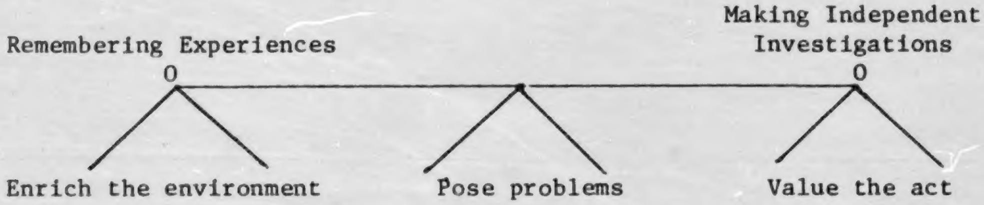
Newborn infants demonstrate that they are learning in many different and mutually supportive styles. We might consider that they constitute a continuum from the more primitive to the more characteristically human modes of learning.

Children filter the overabundance of data supplied by their senses, and they store some of it in their memory banks. I will refer to such storage as "remembering experiences"...the "primitive" end of the continuum. (It has been noted that sticks also learn in this way: you put a notch in a stick and it's there tomorrow.)

But newborn infants also set goals for themselves and employ continuing trial and error until they accomplish those goals -- setting new goals along the way. I will refer to this phenomenon as "making independent investigations"...the more human end of the continuum.

Adults are cast by the nature of things as intervenors in this instinctive learning process. While Rousseau advocated far-reaching nonintervention, most parents and teachers want to intervene as protectively as they can. While Piaget carefully avoids intervening lest he "contaminate" the natural learning process he wants to observe and report, Bruner searches for ways we can actively intervene in that process, to maximize the independent investigative efforts of learners. It is with deep respect for and appreciation of these and other seminal thinkers in "learning theory" that I suggest the following basic strategy.

The diagram below is intended to suggest the continuum from "primitive" to essentially "human" kinds of thinking. I would like to consider here different kinds of intervention.



"Enriching the environment" is a familiar expression meant to include a wide range of activities -- from introducing attractive and responsive toys, to insuring that other human beings of varying ages are available for human interaction.

Intervention at the other end of the continuum -- "making independent investigations" -- cannot be contrived; this kind of activity depends entirely on the learner's decision to work toward a personally selected goal. However, adults can find ways to let learners know how highly they value such activity -- after the act.

"Posing problems" is intervening somewhere nearer the midpoint of the continuum as suggested by this diagram.

At the "primitive" end of the scale the learner is something like a sponge -- soaking up selected data for the memory banks. At the other end, learning is a self-initiated, self-contained, independent activity. "Posing problems" is a combination of external and internal factors -- the problems are presented by adults, textbooks, other children, environmental situations -- outside factors. But the solutions depend on each individual learner.

These three kinds of learning -- remembering experiences, solving problems, and making independent investigations have been somewhat lightly referred to as (1) learning, (2) learning to learn, and (3) learning to learn to learn.

It is my own personal bias and faith that involving children in problem solving will create a climate that is most conducive to learners saying "I wonder what if...?" -- and going on to satisfy their curiosities and interests ... setting goals for themselves and working toward them.

Posing problems offers the learner an opportunity to climb into the driver's seat to take over. I believe that once in control and with the foreknowledge that the adults around attach high value to independent exploration, learners are likely to go beyond the problem posed, attracted by unexpected regularities, or responding to whims, or casting the problem in another context, to test the generalities of newly found relationships, or searching for more elegant solutions for the problem. When they are in the driver's seat, children have demonstrated an awesome capacity to learn anything they want to learn.

There are two effective tactics for direct intervention in the learning process, to help maximize the likelihood that children will set new goals and reach them: (1) enriching the environment and (2) posing problems. And there is an indirect kind of intervention: letting children know that grown-ups place the highest value on "independent investigations...by "stroking" the investigators and by "modeling the activity."

So much for theory. What about mathematical content? What mathematical "skills and learning" do we value highly? What are the goals of our intervention in the area? How can we intervene most productively (as our own values persuade us)?

Mathematics is, in part, concerned with ways to write down and talk about some aspects of what happens when you move things around in different ways -- notational schemes that help people communicate about what they do.

One specialized bit of mathematics is concerned with tactics and strategies that are useful in avoiding one-by-one counting -- computational skills.

But in a more inclusive sense, mathematics is the search for relationships (often numerical and spatial) that are not obvious until we dig below surface appearances. Solving problems and making independent investigations in mathematics call for the skills and attitude we esteem most highly.

It was this broader view of goals in mathematics that led the contributors to the "Cambridge Report" to value situations in which children would say "I don't know, but I'm going to try to find out." The authors said they knew of a few such situations -- but many more were needed.

Actually such situations abound in mathematics. Enough situations are accessible to elementary school children to flesh out a complete curriculum. And there are enough interesting problems involving "computations" and other "basic skills" to provide for more than adequate development in the areas SATs are designed to monitor.

Initial efforts in this direction have had dramatic results both in terms of SAT scores and more positive attitudes (of children and teachers) toward mathematics.

Finally, pioneering efforts are underway to develop techniques for gaining insight into ways individual children behave when they are confronted with substantial problems that involve mathematics -- monitoring growth and development in the affective domain. (See "Let's Pull Together" which I have written as a sequel to this paper.)

Some Direct Answers to Questions Posed

What Are Some of the Important Mathematical Skills and Learning?

1. Understanding from the beginning that arithmetic is simply a socially acceptable way to record certain aspects of events and outcomes.

2. Understanding that "arithmetic computation" is a never ending search for more productive ways to avoid one-by-one counting.

3. Understanding at the outset that mathematics is essentially a problem-solving endeavor -- a search for relationships that are not obvious -- and that computational facility is one of the tools of the trade.

4. Adopting an initial plan of attack on problems as they arise and looking for efficient ways to record results.

5. A wondering attitude toward solutions and records. Could they yield deeper insight?

6. Selecting problems to work at or goals to reach that are appealing and that somehow seem to warrant an effort.

7. Being sensitive to similarities between problems that originally seemed unrelated.

What Are Some of the Obstacles to Developing Proficiency in These "Skills and Learning?"

1. Most school arithmetic ignores children's demonstrated capacity to learn what they want to learn.

2. Most school mathematics programs are designed to teach children "concepts" they already know.

3. Traditional scope and sequence in elementary school mathematics shows complete disregard for what we know about ways young children learn.

4. Children are given almost no opportunity to make choices and to participate in setting goals.

5. Trial and error is discouraged. Central concern is focused on committing [illegible]e right answer" to memory -- to be automatically recalled on command.

6. Children are given few opportunities and given little encouragement to solve problems: They are exhorted to engage in more "drill and practice" that considers a host of unrelated, irrelevant examples of bits to memorize.

7. The normative and criterion-referenced tests that are concerned only with development of computational skills, in effect, tell children and teachers what goals have been selected as important.

8. Educational agencies are reluctant to provide appropriate materials teachers need to change what happens in the classroom, and the professional development teachers need in using new material more productively.

9. The model children are shown of mathematics in the lives of adults -- parents, teachers, and community -- is far from attractive.

What Might NIE Do To Help?

1. Undertake a widespread comparison of the results achieved by skill-oriented management systems in terms of SAT scores and criterion-referenced tests, and results using the same instruments of more humanistic and problem-solving programs. (Note: From a humanistic viewpoint, this is letting the adversary choose the weapons.)

2. Support efforts to find out how much mathematics, as measured by SATs, children would learn if they were given no formal instruction. (More modestly, experimental groups given a complete year's vacation from all "story problems" or "applications," and given the same vacation but opportunities to create

their own "story problems" and "applications," could be compared with control groups having traditional instruction.)

3. Support efforts to develop more effective methods for monitoring development of problem-solving skills and attitudes, and popularize the existence of such methods.

4. Summon the most creative behaviorists and humanists to work together under an elongated slogan: "An individualized diagnostic/prescriptive approach to learning -- at the problem-solving level."

5. Initiate an effort to let the public see that what they recognize as school mathematics could be recast in terms of interesting problems available to everyone -- a Project Two, perhaps.

NEWTON GOES TO EUCLID

Jerrold Zacharias and Mitchell Lazarus

What are Basic Mathematical Skills and Learning?

The title question is not the best place to begin. First we should be asking, "Why are we teaching mathematics to young children at all?" Obviously, a choice of "basic" skills ought to depend on the reasons for teaching them. Yet the two of us have heard and read far less discussion of why than of what.

For us and our colleagues, mathematics is important to children for its usefulness in childhood and later life. Parts of mathematics can help people understand their society, their day-to-day environment (including much of its technology), the public issues, and their own personal affairs. The kinds of mathematics we teach should reflect what we think students can use best.

Notice what this leaves out. We are not training future mathematicians, for addressing the present many to reach the future few is terribly inefficient. We are not "improving children's minds" in some abstract, mystical way, although we are improving their coping skills with specific mathematical tools. And we set aside any tradition that says that the "basic skills" important in the past must, for that reason alone, still be basic today.

We believe in teaching mathematics so that children can use it to better their own lives. We understand how difficult this can be to accomplish. But in the end, our client is the child, and the adult the child will become.

With that, we can turn to the what -- to the "skills and learning" that will serve the child best through his or her life. Any such list of topics will generate much discussion; we have no delusions about setting down the final word here. But, the following list of topics is one place to begin thinking about the kinds of mathematics people can use best.

For example, people measure. This is how we translate reality into numbers. Most of us can use measuring instruments, handle units, and understand precision (or its lack) in our measurings. We know intuitively that measurement is an abstraction from reality, and we work successfully with the results of measurement.

Most people gain these skills and understandings the hard way, for the topic seldom comes up in school. Engineers and scientists learn to measure properly in college, plumbers and carpenters learn on the job, and the rest of us fumble for years with rulers and scales before we get it straight. But measurement is important and useful in mathematics. We should be giving it far more attention in school.

Similarly, estimation plays a major role in real, everyday uses of mathematics. Most people probably estimate several times in a day. (That, too, is an estimation!) Teachable estimation skills can help children acquire a good sense of "scale" in numbers and in measurement units. Estimating successfully can help, we think, to break down the fear of numbers many people show. And learning to estimate puts to rest the common, mistaken belief that mathematics must always be an "exact science" in which each problem has only one right answer.

Parallel to estimation is approximation -- arriving at rough answers in arithmetic quickly. Since most real life problems need only approximate answers, these methods can save a great deal of time, energy, and frustration. Like estimation, approximation can arouse easier, more comfortable feelings about mathematics than come from detailed arithmetic.

Ratio and proportion are important topics that often arise in realistic problems. In the abstract, they are notoriously difficult for young children. But we think many students can handle the idea of scale, as in maps and scale models, much more easily. Using maps and models can form a natural and straightforward approach to ratio and proportion. And scale is a valuable concept in itself, along with the skills that apply to maps and models as tools in their own right.

Perhaps the most useful collection of topics in all of applied mathematics, from the elementary level to very advanced work, deals with rates of change. We are not proposing formal calculus for young children, but we do suggest materials -- especially hands-on activities -- focusing on how fast things change. This includes ways to record, measure, and calculate rates of change. How fast something is changing is usually just as important as any other measure -- a point many adults, including well-educated people, do not fully appreciate. Teaching this idea to children can help them better understand many kinds of situations. And in the context of rates, the topic of graphing will come up naturally and easily.

Finally, the notion of equilibrium -- both static and dynamic equilibrium -- is basic to a great many natural and manmade processes, ranging from keeping a kite in the air to the results of the population explosion. Mathematically, equilibrium arises from several rates interacting in certain ways. Even so, some aspects of equilibrium can doubtless be made clear to even very young children. Other ideas in equilibrium depend more directly on a good understanding of rates of change, and are better suited to later parts of the curriculum.

There are other topics we would include also: number line; geometry with instruments; some elements of algebra, especially the basics of variables, functions, and "unknowns"; and some ideas in probability and statistics.

Notice that this rough outline omits two areas that now form a large part of most elementary mathematics education.

First, we are leaving out calculation, except for simple arithmetic on numbers of one or two digits. Few adults nowadays use most of the calculation we teach in school, especially long division, long multiplication, and complicated work on fractions. With sophisticated cash registers, computers, and small calculators available nearly everywhere, hardly anyone calculates by hand anymore, except in school.

Inevitably, some day, we shall no longer teach longhand calculation in school either. The skill is simply no longer worth the tremendous effort needed to acquire it. And we cannot be surprised that most students dislike "mathematics" after 6 years or so of arithmetic. Deemphasizing calculation now, instead of later, will spare several generations of students the tedious drudgery of longhand arithmetic, and the "mathophobia" that often results.

Still, children must learn enough arithmetic to understand how numbers interrelate, and to appreciate what arithmetic operations mean in terms of real situations.

Second, we can leave out special "new math" topics like set theory and non-decimal number base. These are pointless to most people, and especially to children trying to figure out the number system. Although they are certainly important to specialists, this does not warrant them a place in the general curriculum.

Omitting these topics, we still retain nearly all the math that most people use when they use mathematics in realistic situations -- what some of us have begun to call the "mathematical arts."

As this new approach begins to take shape and direction, we are realizing that its styles of instruction must depart from the conventional classroom materials and techniques. The most common teaching media -- the printed page and the chalkboard -- will probably not be adequate for the content we propose.

Primarily, the new curriculum will need "hands-on" activities in the classroom. In teaching the relationships between mathematics and the real world, it makes sense to bring pieces of the world into the classroom, in the form of activities. Audiovisual media, especially television and film, can become important components for parallel reasons. Like the activities, these can bring valuable experiences into the classroom, and can illustrate clearly and dramatically how mathematics works in real settings.

Another medium will be important also: calculators, slide rules, abacuses, nomograms, and further aids to calculation. Rote calculation should not dominate the mathematical content. But the need for calculation remains nonetheless, making mechanical help necessary. Moreover, children with calculators can mathematically explore more complete and realistic situations than children without, adding to the breadth and depth of educational experience.

Calculators may also help children understand some of the properties in simple arithmetic. "Experimenting" with the machine may give some insight into the operations it carries out. Making this approach work well in practice is a research issue -- one that has drawn surprisingly little attention, considering the impressive educational potential in these little machines and the tremendous impact they are having on how adults do arithmetic.

Still, our main emphasis on calculators is not to aid in the teaching of calculation, but to make complicated longhand arithmetic unnecesary in the classroom, much as it is becoming unnecessary elsewhere.

The curriculum stemming from this approach to mathematics promises to be appealing in several ways. Children will find it more enjoyable. And teachers should find it more appropriate than many of the curriculums now in use. It should also be more effective and more lasting in its effects.

Most important, this is mathematics that students can use to good advantage. But further, it is the kind of mathematics that changes and sharpens one's perceptions of the everyday world. Rising inflation becomes a disequilibrium among several rates of change, and more easily understood. The seemingly trivial 1.5 percent a month charged by a credit card shows itself to be a potentially draining proposition. And many of the numbers that politicians and advertisers brandish about become suspect, in some detail.

This is, after all, the main goal of education: to equip the child for the present and for later life with more accurate perceptions and improved skills. The mathematics we are teaching in schools is not doing that job, in view of people's present and future mathematical needs. It is time to reconsider and then to change.

What are the Major Problems, and What Role Should NIE Play?

The major problem in mathematics education is that we are teaching the wrong mathematics. This was not always true; once longhand arithmetic was a basic skill for employment and for personal management. The "new math" too began with the best motivations, and was a worthwhile experiment. But those days will not return, and as educators we must respond to the changes in our clients' needs.

On the other side, the current mathematics problem does not arise primarily from inadequate teaching, or from poor materials, or from any systematic decline of ability among students. Certainly, these are factors in particular situations. But from a national standpoint, we are simply teaching content that evolving needs have made inappropriate. Most students have a good general sense of what is important, and we can hardly fault them for not wanting to bother with present elementary mathematics.

If our perception of the problem is correct, then the remedy, at least in principle, is simple: Change the content.

We described earlier the content we think is most appropriate for the coming decades, fully aware that any national consensus will require a great deal of give-and-take discussion. NIE can play a pivotal role in helping this discussion begin.

Second, content must become curriculum. This involves two large steps: outlining a curriculum, in detail and with examples, from a set of end-point objectives; and then designing, trying out, and publising the curriculum materials themselves.

Third, all concerned must appreciate that standardized achievement tests have a stranglehold on meaningful curriculum change. It is impossible to change curriculum without changing the tests at the same time. With the old tests in use, a new curriculum stays on the shelves -- because no school system can afford to look poor in the national averages, no matter how much it prefers a new curriculum. This is the worst kind of chicken-and-egg situation, because neither curriculum publishers nor test publishers can move first. Only through the intervention of a third kind of agency, such as NIE, can the problem ever become unstuck.

Little that we propose here is completely new. Indeed, our published articles have drawn much mail saying, in effect, "I'm doing that, and it works!" But there is no set of curriculum materials presently on the market that comes even close to addressing the concerns we raise here. Until someone makes a decisive move toward improving the content, the elementary mathematics situation will continue in its present steady decline.

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**Although E.G. Begle contributed a paper to the conference, he was unable to attend the meeting itself.